A NEW ALGEBRA

VOL. I

BARNARD AND CHILD

WITH ANSWERS

A NEW ALGEBRA

"J'ai horreur d'un enseignement qui n'est pas toujours sincère : le respect de la vérité est la première leçon morale, sinon la seule, qu'on puisse tirer de l'étude des sciences."

Tannery, Leçons d'algèbre et d'analyse.

MACMILLAN AND CO., Limited
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
ATLANTA · SAN FRANCISCO

THE MACMILLAN CO. OF CANADA, Ltd.
TORONTO

A
NEW ALGEBRA

BY

S. BARNARD, M.A.

ASSISTANT MASTER AT RUGBY SCHOOL, LATE FELLOW AND LECTURER AT EMMANUEL COLLEGE, CAMBRIDGE

AND

J. M. CHILD, B.A., B.Sc.

LECTURER IN MATHEMATICS, TECHNICAL COLLEGE, DERBY ; FORMERLY SCHOLAR AT JESUS COLLEGE, CAMBRIDGE

VOLUME I.

CONTAINING PARTS I. II. AND III.

WITH ANSWERS

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1908

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO. LTD.

PREFACE

THIS book represents an attempt to meet the growing demand for a School Algebra which contains a logical development of the subject in accordance with modern views.

The chief difficulty has been to produce a book which satisfies this condition and is at the same time practically useful for school purposes. After many arrangements and re-arrangements of the material, the book has taken a form which may be described briefly as follows:

This first volume is divided into three parts. Part 1 is a generalized Arithmetic in which letters are employed to represent Natural Numbers, and the idea of Algebraic Form is introduced. In Parts 2 and 3, Zero and Negative Numbers and Fractions are considered. These two new classes of numbers are defined so that the expressions $a-b$ and a/b may always have a meaning. For each of the new classes of numbers:

(i) Definitions are given relating to equality and inequality.

(ii) The fundamental operations are defined in accordance with the principle of Permanence of Form.

(iii) The meaning of the numbers in relation to measurement is considered.

Negative Numbers appear for the first time on p. 149. An attempt was made to follow the recommendations of the Mathematical Association and to introduce Negative Numbers at a much earlier stage, but with this arrangement it was found that the exercises possessed little variety, and that the ex-

planations were unsatisfactory. Similar considerations led to the appearance of the Distributive Law in Chapter II. as well as in Chapter V.

An effort has been made to improve upon the loose and unsatisfactory treatment usually accorded to Equations; easy problems and equations on which they depend have been introduced at the very beginning.

By relegating "Long Division" to Chapters XVI. and XIX., it has been possible to point out the true significance of the Division Transformation.

The present volume will be followed by a second, which will complete the usual school course up to the Binomial Theorem.

It was considered undesirable for the student to be introduced to any process until he is able to understand the justification for it; the "long rule for H.C.F." and the historic method of finding square root are therefore deferred to Vol. II.

It is not proposed that the beginner should memorize the bookwork in such a way as to be able to reproduce set pieces, but that he should learn to answer questions on the subject matter. The pupil who desires mere dexterity in manipulation will be able to acquire it by working through the exercises.

The authors have derived their ideas as to Number from the writings of Cantor, Chrystal, Clifford, Dedekind, Hobson and Tannery; and they are glad to acknowledge their deep obligation to Mr. R. B. Henderson and the Rev. D. E. Shorto (both of Rugby School) for many valuable suggestions, and particularly to Mr. Henderson for his careful reading of the proof sheets and for his very able criticism. Thanks are also gladly expressed to Professor R. A. Gregory and Mr. A. T. Simmons for much valuable advice throughout the preparation of the book, and for assistance in reading the proof sheets.

A large number of the examples are original: the others have been taken from recent papers set in examinations for the certificates of the College of Preceptors, Oxford and Cambridge Locals, London University Matriculation, Board of

Education, Intermediate Board for Ireland, and Scotch Leaving Certificates. For permission to publish these questions, thanks are due to the Controller of His Majesty's Stationery Office, the Senate of the University of London, the Delegates of the Oxford Local Examinations, the Cambridge Local Examinations Syndicate, the College of Preceptors, and the Intermediate Board for Ireland.

S. BARNARD.
J. M. CHILD.

August 1908.

CONTENTS

PART I. NATURAL NUMBERS.

PART II. ZERO AND NEGATIVE NUMBERS.

PART III. RATIONAL NUMBERS.

VOLUME I.

CONTAINING PARTS I. II. AND III.

A NEW ALGEBRA

PART I. NATURAL NUMBERS

CHAPTER I.

THE LAWS OF COUNTING.

1. Arithmetic and Algebra are both parts of the **Science of Number.** The signs denoting the fundamental operations of Arithmetic are

$+$ read *plus*, and called the sign of addition;

$-$ read *minus*, and called the sign of subtraction;

$\times$ and $\div$ called respectively the signs of multiplication and division.

In Algebra, these signs receive an extended meaning and, in consequence, we are led to new classes of numbers, not considered in Arithmetic; we therefore begin by considering the exact nature of the fundamental operations.

NOTE. 'Fundamental' means *at the bottom.*

2. Counting. To find the number of things in a group, or collection of things, we count them. Taking the things, one by one, in any order we choose, we count one, two, three, ... ; or we label the things with the symbols 1, 2, 3, ..., which in this process are always used in the same order.

Finally, we say that the last of the symbols used denotes the number of things in the group.

NOTE. ...... is an abbreviation for *and so on.*

In this process, it is important to observe two points:

(1) The result has nothing whatever to do with the character of the things counted;

(2) The order in which the things are counted does not affect the result: thus,

Number is independent of the character of the things counted and of the order of counting.

On this **Law of Counting** the whole Science of Number is based.

NOTE (i). A 'symbol' is that which *represents* something.

(ii) In Mathematics a Law lays down some essential principle of universal application.

The symbols 1, 2, 3, 4, 5, ... are called **numbers** or **integers**; they are also called the **natural numbers** or **whole numbers.** The number 1 is often called **unity.** The symbols, standing in the above order, form the **natural scale.**

The sign $=$, called the **sign of equality**, means *stands for*, or *is the same number as*, or *is equal to.*

ABBREVIATIONS. $\therefore$ means *therefore*; $\because$ means *because.*

3. Addition. The result of adding 3 to 4 is denoted symbolically by $4+3$, and is defined as the number *which occupies the third place after* 4 *in the natural scale.*

Thus to find the number denoted by $4+3$, we start with the number 4 on the natural scale, and count three *more* (5, 6, 7); the last number (7) used in this process is the number denoted by $4+3$.

Again $2+3+4$ denotes the number obtained by adding 3 to 2 and then adding 4 to the result, the operations being performed in order from left to right.

The process of counting by which the number denoted by $2+3+4$ is obtained may be represented as follows:

$$1,\ 2;\ 3,\ 4,\ 5;\ 6,\ 7,\ 8,\ 9.$$

The last symbol used is 9, and we write $2+3+4=9$.

Ex. 1. **Prove that $4+3=3+4$.**

By definition, $4+3$ is the third number after 4 in the scale, and denotes the last number reached in the following process of counting:

$$1,\ 2,\ 3,\ 4;\ 5,\ 6,\ 7.$$

Again, $3+4$ denotes the last number used in the process of counting represented by

$$1,\ 2,\ 3;\ 4,\ 5,\ 6,\ 7.$$

The last number is the same in both cases;

$$\therefore\ 4+3=3+4.$$

Ex. 2. **Prove that $2+3+4=4+3+2$.**

[A similar method to that used in Ex. 1 applies.]

The process of finding the number denoted by $2+3+4$ is called *adding* 2, 3 *and* 4 *together.*

Def. The result of adding several numbers together is called their **sum.**

4. Representation of Numbers by Letters. In Algebra, numbers are often represented by **letters**: reasons for this mode of representation will be given later.

The **value** of a letter, used to represent number, is the number for which the letter stands. To say "Let $x=4$" means that, in some particular piece of work, x is to stand for 4. To say "$a=b$" is to assert that a and b stand for the same number: in this case, it is convenient to call a and b **equal numbers.**

The statement "$a=b$" is called an **equality or equation**: a is called the *left hand side* and b the *right hand side* of the equality.

If a and b stand for two numbers, a is said to be **greater than** b or **less than** b according as a follows b or as a precedes b on the natural scale.

Other ways of stating the equality $4+3=7$ are as follows:

(1) 7 is greater than (or exceeds) 4 by 3;

(2) 4 is less than 7 by 3.

Abbreviations. $>$ means *is greater than*; $<$ means *is less than.*

The statements, "$a>b$," "$x<y$" are called **inequalities.**

Def. The symbol $3x$ is an abbreviation for $x+x+x$.

Thus, if $x=4$, then $3x$ stands for $4+4+4$, that is for 3 *times* 4, or 12.

Ex. 1. *If* $x=6a+3b$, *find the value of* x *when* $a=1$ *and* $b=5$.

Here, $x=(6 \textit{ times } 1)+(3 \textit{ times } 5)=6+15=21$.

This example shows that, if the values of a and b are known, the equation $x=6a+3b$ enables us to find the value of x. This equation is therefore said to *determine* the value of x in terms of the values of a and b.

An equation which determines the value of one letter in terms of the values of other letters is often called a **formula.**

Ex. 2. *A purse contains* x *pounds,* y *shillings and* z *pence. Express, in pence, the value of the contents of the purse.*

£1 = 240 pence, £x = (240 times x) pence = $240x$ pence.

In the same way, y shillings $=12y$ pence.

Hence the value in pence of the contents of the purse

$$=240x+12y+z.$$

Ex. 3. *Name the three consecutive numbers of which* x *is the least.*

If x stands for any number, the number next to x and greater than x is $x+1$. The required numbers are, therefore,

$$x,\ x+1,\ x+2.$$

Def. A collection of symbols denoting numbers and operations to be performed on them is called an **algebraical expression.**

$7a$, $8b$ and $5c$ are called the **terms** of the expression $7a+8b+5c$, and 7, 8 and 5 are called the **coefficients** of a, b, c respectively in the expression.

EXERCISE I. MENTAL.

Note. *In exercises called* 'Mental' *the answers may be given* viva-voce, *or they may be* written down (*no actual work on paper being necessary*). *The student is advised to work through the whole of these mental exercises.*

If $x=7$, find the value of

1. $5x$. **2.** $x+5$. **3.** $5+x$. **4.** $2x+3$.

5. $2+3x$. **6.** $2x+3x$.

If $x=1$ and $y=2$, find the value of

7. $x+y$. 8. $2x+y$. 9. $x+2y$.

Find the value of $x+y$ when

10. $x=3$ and $y=4$. 11. $x=4$ and $y=3$.

Find the number which exceeds

12. 20 by 7. 13. 20 by x. 14. x by 20. 15. x by y.

In the equation $s=a+b$,

16. If $a=5$ and $b=3$, what is the value of s?

17. If $a=2$ and $s=6$, what is the value of b?

18. If $b=3$ and $s=7$, what is the value of a?

What is the total day's walk if the distances covered in the morning, afternoon and evening are as follows:

19. Morning 10 miles, afternoon 6 miles, evening x miles?

20. Morning x miles, afternoon y miles, evening 4 miles?

21. Morning x miles, afternoon y miles, evening z miles?

22. Morning $3x$ miles, afternoon $2x$ miles, evening x miles?

23. If six times a certain number is 30, what is the number?

24. If $6x=30$, what is x?

25. If in Ex. 22, the total day's walk was 30 miles, how far was walked in the evening?

26. Express symbolically (*i.e.* by using signs) the statement "20 is the sum of 17 and 3."

27. If s is the sum of a and b, what is the formula giving s in terms of a and b?

28. Express a yards (i) in feet, (ii) in inches.

29. Express x miles in yards.

30. Express £x in (i) half-crowns, (ii) florins, (iii) sixpences.

How many pence are there in a purse containing

31. x shillings and y sixpences? 32. x half-crowns and 2 shillings?

A man has three boxes of eggs and each box contains a eggs:

33. How many eggs has he altogether?

34. If $a=12$, how many eggs has he?

35. If the total number of eggs is 48, what is the value of a?

Two bags contain apples; the first has a apples, the second has b apples more than the first.

36. How many apples are there in the second bag?

37. How many apples are there altogether?

38. If x is the total number of apples, what is the equation connecting a, b and x?

A's age is now a years.

39. How old will he be 6 years hence?

40. How old will he be y years hence?

41. In how many years will his age be $a+x$ years?

Seven years ago A's age was x years.

42. What is his present age?

43. What will his age be in 3 years?

44. What will his age be in y years?

45. A is twice as old as B, and B's age is now x years. How old will A be in y years?

46. What are the three consecutive numbers of which $x+2$ is the least?

47. Find the sum of the two consecutive numbers of which x is the least.

48. What is the even number next above 20?

If n stands for any number,

49. What kind of number is $2n$? 50. What kind of number is $2n+1$?

51. Name three consecutive even numbers of which $2n$ is the least.

52. Name three consecutive odd numbers of which $2n+1$ is the least.

53. If $5n+2=22$, what is the value of $5n$? What is the value of n?

54. If n stands for a certain number, state in words the problem the solution of which is contained in Ex. 53.

55. A clerk's salary is £50 for the first year, and for every subsequent year he gets a rise of £x. What is his salary for the second year? What for the eleventh year?

56. If in Ex. 55 the clerk's salary is £145 for the twentieth year of his service, what is the value of x?

5. Multiplication. The process of adding several *equal* numbers together is called **multiplication**. The number $4+4+4$ is called the **product** of 4 by 3 or 4 multiplied by 3, and is written in either of the forms 4×3 or $4 \,.\, 3$. Here 4 is called the **multiplicand** and 3 is called the **multiplier**.

NOTE. To avoid confusion, the dot, when used as a sign of multiplication, is placed as low as possible. Thus $3 \,.\, 4$ means 3×4 and $3{\cdot}4$ means 3 *point* 4, 3 *decimal* 4 or $3\frac{4}{10}$.

Ex. 1. **Prove that $4 \times 3 = 3 \times 4$.**

Take 3 rows of stars, each row containing 4 stars, and arrange them so that the stars are in 4 vertical columns. (Fig. 1.)

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}$$

FIG. 1.

The number of stars in the 3 rows of 4 stars each is 4×3,

and the number of stars in the 4 columns of 3 stars each is 3×4.

Since number is independent of the nature of the things counted and of the order of counting, it follows that $4 \times 3 = 3 \times 4$.

DEF. The sum of b equal numbers, each of which is represented by a, is called the **product of** a **by** b or a **multiplied by** b, and is denoted by any of the expressions ab, $a \,.\, b$, or $a \times b$.

Thus $ab = a \,.\, b = a \times b = a + a + a + \ldots$ until there are b a's.

When one number is multiplied by another, the numbers are said to be "multiplied together," and the result is often called "the product of the numbers."

DEF. The numbers a and b are called **factors** of the product ab.

In the last example it has been shown that $4 \times 3 = 3 \times 4$, and the argument *does not* depend on the *particular numbers chosen*, hence we conclude that

The product of two factors is independent of the order of the factors.

In other words, if a and b stand for any two of the natural numbers, then

$$\mathbf{ab = ba.}$$

We shall nevertheless find it necessary sometimes to mark the *order* of the factors of a product. When this is the case, the definitions already given will be adhered to, so that

(1) in the case of a product like ab, where *both* the factors are represented by *letters*, ab will mean the same as $a \times b$;

(2) in the case of a product like $3x$, where *one* factor is expressed *numerically*, $3x$ will mean 3 *times* x or $x \times 3$.

Def. The expressions abc, $a \,.\, b \,.\, c$, $a \times b \times c$ all stand for the same number. This number is obtained by multiplying a by b and then multiplying the result by c, and is called the **continued product** or simply the **product** of the numbers a, b, c. The numbers a, b, c are called **factors** of the product abc.

In the case of a product like $3ab$, where *one* of the factors is expressed numerically, $3ab$ will stand for 3 *times* ab, that is for $ab + ab + ab$.

Def. $x \times x$ is called the **square** of x or x **squared**, and is written $\mathbf{x^2}$; $x \times x \times x$ is called the **cube** of x or x **cubed**, and is written $\mathbf{x^3}$.

The expression x^2y^3 stands for "the square of x multiplied by the cube of y."

Ex. 2. *If* $x = 3$ *and* $y = 2$, *find the value of* x^2y^3.

$$x^2y^3 = 3^2 \,.\, 2^3 = 9 \,.\, 8 = 72.$$

Ex. 3. *If* $x = 3$, *find the value of* $2x^3 + x^2 + 5x$.

$$\begin{aligned} 2x^3 + x^2 + 5x &= 2 \,.\, 3^3 + 3^2 + 5 \,.\, 3 \\ &= 2 \,.\, 27 + 9 + 15 \\ &= 54 + 9 + 15 \\ &= 78. \end{aligned}$$

6. Representation of Numbers by Letters. The following are illustrations of the different ways in which letters are used to represent numbers:

(1) The general theorem of which the equality $4 + 3 = 3 + 4$ is an instance is as follows: "If a and b stand for *any* two numbers, then $a + b = b + a$."

(2) In a piece of work where frequent use is made of some particular number, say of 1728, it may be convenient to give this

number another *name* and to call it a. In this case we should say, "Let $a = 1728$," meaning that a is to stand for the number 1728.

(3) The equation $x + 3 = 7$ is an algebraical statement of the question, "To what number must 3 be added that the result may be 7?"

Here, x stands for a number which is said to be **unknown**, although its value (namely 4) is determined by the given equation.

7. Equations and Identities. When x stands for 4, the two sides of the equation $x + 3 = 7$ are equal; the equation is therefore said to be **satisfied**, and the value 4 of x is called a **solution.**

In this case, 4 is the *only* solution, because 4 is the only number to which if 3 is added, the result is 7.

Again, if $3x = 6$, then x must stand for 2, for 2 is the only number which, if multiplied by 3, produces 6. Hence 2 is the solution of the equation $3x = 6$, and there is no other solution.

DEF. A statement of equality which is true for all values of the letters concerned is called an **identity**.

Thus, $a + b = b + a$ is an *identity*, for it is true for all values of a and b.

DEF. A statement of equality which is true only when the letter or letters concerned have some particular value or values is called an **equation**.

Ex. 1. *Show that the values 5 and 6 of x are solutions of the equation $x^2 + 30 = 11x$.*

(i) When $x = 5$,

$$x^2 + 30 = 5^2 + 30 = 25 + 30 = 55$$
$$\text{and } 11x = 11 \times 5 = 55;$$
$$\therefore x^2 + 30 = 11x.$$

(ii) When $x = 6$,

$$x^2 + 30 = 6^2 + 30 = 36 + 30 = 66$$
$$\text{and } 11x = 11 \times 6 = 66;$$
$$\therefore x^2 + 30 = 11x.$$

Hence the given equation is satisfied if $x = 5$ or if $x = 6$, so that the values 5 and 6 of x are solutions of the given equation.

EXERCISE II. MENTAL 1–30.

For what do the following stand (when written at full length)?

1. $2x$. 2. x^2. 3. $3x$. 4. x^3. 5. $2x^3$. 6. $3x^2$.

State the values of the following (i) when $x=1$, (ii) when $x=2$.

7. $2x$. 8. x^2. 9. $3x$. 10. x^3. 11. $2x^3$. 12. $3x^2$.

13. Express $2ab$ as a sum.
14. How is the number denoted by abc to be found?

If $a=1$, $b=2$, $c=3$, find the values of

15. abc. 16. bca. 17. cab. 18. $abc+bca+cab$
19. $abc+2bca$.

State the cost of the following:

20. 20 lbs. of tea at x pence per lb.
21. a lbs. of tea at x pence per lb.
22. 1 cwt. of tea at x pence per lb.
23. a cwts. of tea at x pence per lb.
24. a lbs. of tea at x pence per lb. and b lbs. of sugar at y pence per lb.

25. A train travels at the rate of v miles an hour. How far does it go (i) in 3 hours? (ii) in t hours?
26. Running at the uniform rate of v feet per second, a man covers s feet in t seconds. What is the formula giving s in terms of v and t?

27. State algebraically the general theorem of which the equality $4\times3=3\times4$ is an instance.
28. If $5+x$ stands for 11, what number is represented by x?
29. If $12=x+x+x+x$, for what number does x stand?
30. If $15=4x+3$, what is x?

If $x=1$, $y=3$, $z=5$, find the values of

31. x^2+xy+y^2. 32. $x^2+y^2+z^2$. 33. x^2yz. 34. xy^2z.
35. $x^2y^2z^2$. 36. x^3+2x^2+3x+4. 37. x^3+y^3. 38. x^y.
39. y^x. 40. 3^x+2^y. 41. z^y. 42. x^yz^x.

43. Walking at the rate of 4 miles an hour,
(i) how far do I go in x hours?
(ii) if I walk 28 miles in x hours, what is the value of x?

44. It takes x hours to walk from A to B at 4 miles an hour, and y hours to walk back from B to A, by the same road, at 3 miles an hour.
(i) What is the equation connecting x and y?
(ii) If $y=8$, what is the value of x?
(iii) If $x=9$, what is the value of y?

45. Taking the formula $s=vt$ (see Ex. 26):
(i) Find the value of s when $v=88$ and $t=10$.
(ii) Find the value of v when $s=220$ and $t=5$.
(iii) Find the value of t when $s=132$ and $v=22$.

46. A stone falls from rest and moves through s feet in t seconds. It is found that (neglecting the resistance of the air) s and t are connected by the formula $s=16t^2$.
(i) How far does the stone fall in 1, 3, 10 seconds?
(ii) If $s=64$, what is the value of t^2? What is the value of t?
(iii) How long does the stone take to fall 400 feet?

47. The length of a rectangle is l feet and its breadth is b feet. If the area of the rectangle is A square feet, it is found that A, l, b are connected by the formula $A=lb$.
(i) Find A when $l=17$, $b=19$.
(ii) Find b when $A=288$, $l=18$.
(iii) Find l when $A=575$, $b=23$.

48. Express in square feet the areas of the following:
(i) A square whose side is x feet.
(ii) A board x feet long and 1 foot wide.
(iii) A floor x feet long and 7 yards wide.
(iv) The walls of a room whose height is h feet and perimeter (distance round) p feet.
(v) The walls of a room 20 ft. long, 15 feet wide and h ft. high.
(vi) The walls of a room whose length is l feet, breadth b feet, height h feet.

49. If $3x+y=10$, what are the values of y when x has the values 1, 2, 3 in succession?

50. If $3x+2y=21$, what are the values of x when y has the values 3, 6, 9 in succession?

51. Prove that (i) the values 10 and 11 of x are solutions of the equation $x^2+110=21x$.
(ii) The values 7 and 8 of x are solutions of $x^2+56=15x$.
(iii) The values 1, 2, 3 of x are solutions of $x^3+11x=6x^2+6$.
(iv) The values 2, 5, 7 of x are solutions of $x^3+59x=14x^2+70$.

52. Prove that $2x^2+15y^2+13y=11xy+6x+20$ in the following cases: (i) if $x=3$, $y=2$; (ii) if $x=8$, $y=1$; (iii) if $x=8$, $y=4$; (iv) if $x=11$, $y=2$.

53. If x and y are two whole numbers such that $xy=12$, write down all the possible values of x and the corresponding values of y.

54. x and y are two numbers such that $xy=12$ and $x+y=7$. Find the possible values of x and the corresponding values of y.

55. If $3ab=24$, (i) what is the value of ab?
(ii) If in addition to the equation $3ab=24$ it is known that $a=2$, what is the value of b?

56. If $5xy+3=138$, what is the value of xy? If it is also known that the sum of the numbers x and y is 12, what are the possible values of x, and what are the corresponding values of y?

Find two numbers which satisfy the following conditions:

57. Their sum is 11, their product is 30.

58. Their sum is 11, their product is 24.

59. Their sum is 11, their product is 18.

60. If x and y are two natural numbers such that $2x+3y=18$ and $xy=12$, prove that $3x+2y$ may have either of the values 22 or 17.

CHAPTER II.

LAWS OF ADDITION.

8. Laws of Addition. When large numbers are to be added, the process of counting is tedious: we therefore commit to memory the sums of pairs of small numbers and apply the **Laws of Addition.** These are the **Commutative** and the **Associative Laws,** and on them all processes of addition depend.

Ex. 1. **Prove that $2+3+4=4+2+3$.**

$2+3+4$ stands for the number obtained by adding 3 to 2 and then adding 4 to the result; thus $2+3+4$ is the last number used in the process of counting represented by

1, 2; 3, 4, 5; 6, 7, 8, 9.

In the same way $4+2+3$ is the last number used in the process of counting represented by

1, 2, 3, 4; 5, 6; 7, 8, 9.

The last number is the same in both cases,

$$2+3+4=4+2+3.$$

This example is an instance of the **Commutative Law for Addition,** which is as follows:

The sum of a series of numbers is independent of the order in which the additions are performed.

This law is an extension of the Law of Counting (p. 4): for

(i) to count a collection of things *by groups* is merely to count the individual members of one group *before or after* those of another group;

(ii) if we may count single things in any order without affecting the result, we may also count groups of things in any order of groups.

(iii) To add several numbers is equivalent to counting a collection of things by groups.

9. Meaning of a Bracket. When an expression is placed within a **bracket**, the meaning is that the operations necessary for finding the value of the expression are supposed to have been performed, and the expression is supposed to be replaced by the number which denotes its value.

Thus, in the expression $2+(3+4)$, the following order of operations is indicated:—Find the value of $3+4$, add this number to 2.

Ex. 1. **Prove that $2+(3+4)=2+3+4$.**

The number $2+(3+4)$ is obtained by adding $3+4$ to 2, and is therefore the last number used in the following process of counting:

$$1,\ 2;\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9.$$

The number $2+3+4$ is obtained by adding 3 to 2 and then adding 4 to the result, and is therefore the last number used in the process of counting represented by

$$1,\ 2;\ 3,\ 4,\ 5;\ 6,\ 7,\ 8,\ 9.$$

The last number is the same in both cases,

$$\therefore\ 2+(3+4)=2+3+4.$$

The brackets in $(2+3)+(4+5+6)$ denote that the numbers are to be added in **groups**, and the last example is an instance of the **Associative Law of Addition**, which asserts that

The sum of a series of numbers may be found by associating the numbers in groups, adding the numbers in the separate groups and finally adding the results.

This law is a further extension of the Law of Counting; for just as, in counting a collection of things, we may place any number of individuals in a group without affecting the result, so we may form one larger group out of two or more smaller groups.

Illustration. I shall have walked the same distance in the day if I walk 3 miles before breakfast, 4 miles between breakfast and lunch, 5 miles in the afternoon and 6 miles in the evening, or if I walk $(3+4)$ miles before breakfast and $(5+6)$ miles in the afternoon.

10. Arithmetical Progressions. Let a "sequence" of numbers be formed in the following manner:

Take any number a as the first number or "term" of the sequence, form a second term by adding any number d to the first term, form the third term by adding the same number d to the second term; continue this process indefinitely, the terms being formed in succession, each by adding the number d to the preceding term.

The resulting sequence is

$$a,\ (a+d),\ (a+2d),\ (a+3d), \dots \text{ continued indefinitely.}$$

This collection of numbers, standing in the above order, is called the Arithmetical Progression, whose "first term" is a and "common difference" d.

DEF. A **sequence** is a collection of numbers which always stand in the same order: the numbers are called the **terms** of the sequence, and the terms are formed in succession according to some definite law.

DEF. An **Arithmetical Progression** is a sequence whose first term is any number (a), and whose subsequent terms are formed in succession, each by adding the same number (d) to the preceding term.

Ex. 1. *Find the 100th term of the arithmetical progression*

$$a,\ (a+d),\ (a+2d),\ \dots .$$

The 100th term is obtained by adding $99d$ to a, and is therefore $(a+99d)$.

Ex. 2. *Find the 100th odd number.*

The odd numbers 1, 3, 5, 7, ... form an arithmetical progression whose first term is 1 and whose common difference is 2.

The 100th odd number is therefore to be found by adding 2×99 to 1.

$\therefore$ the 100th odd number $= 1 + 2 \times 99 = 1 + 198 = 199$.

Ex. 3. *If the first term of an arithmetical progression is 2 and the 10th term is 29, what is the common difference?*

Let d stand for the common difference.

$$\therefore\ \text{10th term} = 2 + 9d = 29.$$

$$\therefore\ 9d = 27 \text{ and } d = 3.$$

EXERCISE III.

If $a=4$, $b=3$, $c=2$, find the value of

1. $a+2b+c$.
2. $2a+3b+5c$.
3. $2a+b+2c$.
4. $5a+6b+3c$.
5. $a+b+7c$.
6. $a+2b$.
7. $b+3c+5a$.
8. $c+2b+8a$.
9. $18c+a$.
10. $3c+b+4a$.
11. $5a^2+3a+2$.
12. b^3+2b+1.
13. $5c^3+4c^2+3c+2$.
14. $a^3+b^3+c^3$.
15. $a^2+b^2+c^2+bc+ca+ab$.

16. Show that (i) the value 2 of x is a solution of the equation
$$5x^3+4x^2+3x=62,$$
(ii) the value 5 of x is a solution of
$$8x^3+6x^2+x=1155.$$

17. Show that the equation $x^2+35=4y^2+12x+4y$ is satisfied (i) if $x=9$ and $y=1$, (ii) if $x=11$ and $y=2$, (iii) if $x=13$ and $y=3$.

18. If x is the sum of the numbers $2a$, $3b$, $4c$, write down the formula for x in terms of a, b, c.

19. From the formula obtained in Ex. 18, find the values of x in the following cases :—(i) when $a=1$, $b=3$, $c=5$, (ii) when $a=3$, $b=5$, $c=7$.

20. On a week's walking tour, the distances walked are as follows :—First day, a miles ; second day, four miles more than the first ; third day, twice as far as the first ; next three days, b miles each day ; last day, 22 miles. If the total distance walked was x miles, find a formula for x. Also find the value of x if $a=12$, $b=17$.

21. Of a boat's crew, two men weigh a pounds each, four weigh b pounds each, two weigh c pounds each and the coxswain weighs d pounds. If the total weight of the crew is x pounds, find a formula for x.

    From the formula, obtain the total weight of the crew when $a=145$, $b=170$, $c=185$, $d=110$.

22. In an innings of an eleven, one man made a runs, two made b apiece, three made c apiece, one made d, the rest did not score, and there were 5 extras. If the total score was x, write down the equation giving the value of x, and find the total score when $a=23$, $b=18$, $c=9$, $d=4$.

23. A is 6 years older than B, B is 6 years older than C, and in 10 years the sum of their ages will amount to 138. What is A's present age?

24. A man opens a shop on Monday. On Monday his profit is x shillings, on Tuesday it is $2x$ shillings. On each succeeding day for the week the profit is equal to the sum of the profits on all the preceding days. What is his total profit on Saturday night?

If his total profit is £4. 16*s.*, how much did he make on Monday?

25. A party of six men pass through a turnstile, one by one. The first man has y shillings in his pocket, the second has 5 shillings more than the first, the third 5 shillings more than the second, and so on to the sixth. How much have they altogether?

If they have £5. 5*s.* between them, how much has the sixth man?

26. A road between two towns, A and E, forty miles apart, passes through three other towns B, C, D. From A to B (10 miles) the road falls at the rate of a feet per mile; from B to C (15 miles) there is a fall of b feet per mile, and from C to D (5 miles) a fall of c feet per mile. The rest of the road is level. How high is A above E?

If $b=4$, $c=3$, and A is 125 feet above E, what is the value of a?

27. A runs 17 feet per second and B runs 16 feet per second. How far does each run in t seconds? If they start a mile apart and run straight towards one another, in how many seconds do they meet?

28. Find the 12th and the 20th terms of each of the following arithmetical progressions:

(i) $a, (a+d), (a+2d), \ldots.$ (ii) $1, 3, 5, 7, \ldots.$

(iii) $2, 4, 6, 8, \ldots.$ (iv) $2, 5, 8, 11, \ldots.$

29. Find the 1000th odd number.

30. (i) If $a+6d=29$, what is the value of d when $a=5$?

(ii) If the first and seventh terms of an arithmetical progression are respectively 5 and 29, find the second term.

31. Find the 21st term of the arithmetical progressions in which

(i) the 2nd term is 4 and the 5th term is 13,

(ii) the 4th term is 9 and the 6th term is 13,

(iii) the 3rd term is 12 and the 8th term is 37.

11. Use of Brackets. From the explanation given in Art. 9, with regard to the meaning of a bracket, it will be understood that

(1) $a(b+c)$ stands for the number obtained by multiplying a by the sum of b and c;

(2) $(a+b)(x+y)$ stands for the product of the numbers $a+b$ and $x+y$;

(3) $(a+b)^2$ stands for the square of $a+b$, that is for $(a+b)(a+b)$.

Note also that the number $a+bc$ is obtained by adding the product bc to a.

Thus, if $x=3$, $y=4$, $z=5$, then

(i) $xy+z=3.4+5=12+5=17.$

(ii) $x(y+z)=3(4+5)=3.9=27.$

(iii) $(y+z)x=(4+5)3=9.3=27.$

(iv) $y+zx=4+5.3=4+15=19.$

(v) $2(x+y)^2=2.(3+4)^2=2.7^2=2.49=98.$

Ex. 1. *Verify the identity*

$$(x+y+z)^2=x^2+y^2+z^2+2(yz+zx+xy),$$

in the case when $x=1$, $y=2$, $z=3$.

When $x=1$, $y=2$, $z=3$, we have

$$\begin{aligned}(x+y+z)^2&=(1+2+3)^2\\&=6^2=36.\end{aligned}$$

$$\begin{aligned}x^2+y^2+z^2+2(yz+zx+xy)&=1^2+2^2+3^2+2(2.3+3.1+1.2)\\&=1+4+9+2(6+3+2)\\&=14+2.11\\&=14+22=36.\end{aligned}$$

Hence, in this case,

$$(x+y+z)^2=x^2+y^2+z^2+2(yz+zx+xy).$$

NOTE. *In the verification of an identity by substituting special values for the letters, each side should be worked out separately as in* Ex. 1.

12. The Distributive Law. The following examples are instances of a fundamental law of Multiplication called the Distributive Law.

Ex. 1. **Prove that** $2\times(3+4)=2\times3+2\times4$.

$$2\times3+2\times4=(2+2+2)+(2+2+2+2) \quad (\textit{Def. of Multiplication})$$
$$=2+2+2+2+2+2+2 \quad (\textit{Associative Law})$$
$$=2\times(3+4). \quad (\textit{Def. of Multiplication})$$

Ex. 2. **Prove that** $(3+4)\times2=3\times2+4\times2$.

[Since $ab=ba$, this follows from Ex. 1.]

The argument in the proofs of the last two equalities *is general*, and holds good if the numbers 3, 4 and 2 are replaced by any three numbers denoted by a, b and c. We therefore conclude that if a, b, c stand for any three numbers, then

$$\mathbf{(a+b)c=ac+bc} \quad \text{.........(1)}$$
and $$\mathbf{c(a+b)=ca+cb.} \quad \text{.........(2)}$$

In the same way it can be shown that if a, b, c, x stand for any numbers, then

$$\mathbf{(a+b+c)x=ax+bx+cx,} \quad \text{.........(3)}$$

so that to multiply $(a+b+c)$ by x, we multiply each of the numbers a, b, c by x and add the results.

The identities (1), (2), (3) are cases of the **Distributive Law**, and (3) may be stated as follows:

The expression $ax+bx+cx$, whose terms contain the common factor x, is the product of the factors $(a+b+c)$ and x.

To factorize an algebraical expression is to express it as the product of factors.

Ex. 1. *Factorize the following expressions*: (i) $3x+3y$; (ii) x^2+5x; (iii) $x^2(x+2)+x+2$.

(i) Here 3 is a factor of each term, hence

$$3x+3y=3(x+y);$$

(ii) $$x^2+5x=x\,.\,x+5x=(x+5)x;$$

(iii) $$x^2(x+2)+x+2=x^2(x+2)+(x+2)$$
$$=(x^2+1)(x+2).$$

Ex. 2. *Find the value of $33^2+33\times67$.*

$$33^2+33\times67=33(33+67)=33\times100=3300.$$

13. Addition of Algebraical Expressions.

The method of adding algebraical expressions will be understood from the following examples:

Ex. 1. *Prove that $2a+3a=5a$.*

$$\begin{aligned} 2a+3a &= (a+a)+(a+a+a) && \textit{(Def.)} \\ &= a+a+a+a+a && \textit{(Associative Law)} \\ &= 5a. && \textit{(Def.)} \end{aligned}$$

Ex. 2. *Prove that $3a\times 2=6a$.*

$$\begin{aligned} 3a\times 2 &= 3a+3a && \textit{(Def.)} \\ &= (a+a+a)+(a+a+a) && \textit{(Def.)} \\ &= a+a+a+a+a+a && \textit{(Associative Law)} \\ &= 6a. \end{aligned}$$

Ex. 3. *Find the sum of $2a$, $6b$, $3c$, $4b$, c, $7a$.*

$$\begin{aligned} \text{The sum} &= 2a+6b+3c+4b+c+7a \\ &= 2a+7a+6b+4b+3c+c && \textit{(Commutative Law)} \\ &= (2a+7a)+(6b+4b)+(3c+c) && \textit{(Associative Law)} \\ &= 9a+10b+4c. \end{aligned}$$

NOTE 1. When the meaning of the brackets in the third step of the work in Ex. 3 is understood thoroughly, this step may be omitted in practice.

NOTE 2. The expression $9a+10b+4c$ cannot be simplified, unless we know the numbers for which a, b, c stand.

Ex. 4. *Find the sum of $2a+3b$, $4b+2c$, $5a+b+3c$.*

$$\begin{aligned} \text{The sum} &= (2a+3b)+(4b+2c)+(5a+b+3c) \\ &= 2a+3b+4b+2c+5a+b+3c && \textit{(Associative Law)} \\ &= 2a+5a+3b+4b+b+2c+3c && \textit{(Commutative Law)} \\ &= (2a+5a)+(3b+4b+b)+(2c+3c) && \textit{(Associative Law)} \\ &= 7a+8b+5c. \end{aligned}$$

NOTE. When the meaning of the brackets is understood, the first and fourth lines of the work in Ex. 4 may be omitted.

In practice it is sometimes convenient to arrange the work of additions as follows:

$$\begin{array}{r} 2a+3b \\ 4b+2c \\ 5a+b+3c \\ \hline 7a+8b+5c \end{array}$$

In the expressions $2a+3b$, $4b+2c$, $5a+b$, the terms $3b$, $4b$ and b, which contain the same letter, are called **like** terms: the numbers 3, 4 and 1 (understood) are called the coefficients of these terms.

The process exhibited in examples 3, and 4, consists in *collecting the like terms of the expressions to be added, and adding their coefficients.*

EXERCISE IV.

State the values of

1. $2+(3+4)$. **2.** $2(3+4)$. **3.** $2.3+4$. **4.** $2+3.4$.

5. $3.98+3.2$. **6.** $5.11+29.5$. **7.** $17.3+3^2$. **8.** $23^2+23.77$.

When $x=4$, $y=2$, $z=3$, state the values of

9. $x(y+z)$. **10.** $xy+z$. **11.** $(y+z)x$. **12.** $y+zx$.

13. $x+y^2$. **14.** $(x+y)^2$. **15.** $2(x+y)^2$. **16.** $(2x+y)^2$.

Factorize the following expressions:

17. x^2+x. **18.** x^2y+2y. **19.** $bc+ca+3c$.

20. x^3+x^2+x. **21.** $abc+c^3$. **22.** $x(x+y)+y(x+y)$.

23. $(x+y)^2+(x+y)$. **24.** $(x+1)^2+2(x+1)$.

25. $x(x+2y)+y(x+2y)$. **26.** $x(x+2y)+2y(x+2y)$.

Simplify

27. $7a+8a$. **28.** $a+3a$. **29.** $a+5a+10a$. **30.** $a+2a+b$.

31. $a+b+2a+3b$. **32.** $a+2b+3c+4a+5c+6b$. **33.** $(x+2y)+2y$.

34. $(x+2y+3z)+(y+3x+5z)+(z+3y+2x)$.

Find the sum of

35. $2x$, 5, 7. **36.** $3x$, 4, 5, $7x$.

37. x, $2x$, $3x$, $4x$. **38.** $3x+2y$ and $2x+3y$.

39. $3x+4y+5z$ and $x+y+z$. **40.** $x+y+z$ and $z+x+y$.

41. $x+y+2z$, $y+z+2x$ and $z+x+2y$. **42.** $(x+2y)+z$ and $x+(z+2y)$

43. Add together $5x+6y+z$, $z+3y$, $y+6x$.

If $z=x+y$, find z in terms of a, b, c when

44. $x=a+2b$, $y=6a+7b+2c$. **45.** $y=a+b+c$, $x=3a+4b+5c$.

When $x=2$, $y=3$, $z=1$, verify the following identities:

46. $x(y+z)=xy+xz$. **47.** $(y+2)x=yx+2x$.

48. $xy+x^2=x(y+x)$. **49.** $x^2(y+1)+(y+1)=(x^2+1)(y+1)$.

50. $x^2+2+9(x+2)=(x+4)(x+5)$.

51. $2(1+3y^2)+7y=(2+3y)(2y+1)$.

52. $x^2+x(y+z)+yz=(x+y)(x+z)$.

53. $(y+z)^2+(z+x)^2+(x+y)^2=(x+y+z)^2+x^2+y^2+z^2$.

54. $x(y+z)^2+y(z+x)^2+z(x+y)^2=(x+y+z)(yz+zx+xy)+3xyz$.

55. $(x+1)(x+2)+(x+2)(x+3)+(x+3)(x+1)+1=3(x+2)^2$.

56. Express symbolically (*i.e.* by using signs)

(i) The result of adding a to the product of b and c.

(ii) The product of the sum of a and b and the sum of c and d.

57. If $3(x+y)=15$, what is the value of $x+y$? Also, if $x=2$, what is y?

58. Find x from the equation $6(x+1)=42$.

59. If $3(x+2y)=27$ and $x=7y$, find the values of x and y.

60. A stone is thrown vertically downwards with a velocity of u feet per second: at the end of t seconds, its velocity is v feet per second and it has fallen through s feet. It is found that u, v, t, s are connected by the formulae

$$v=u+32t, \qquad s=ut+16t^2.$$

(i) If $u=15$ and $t=5$, find v and s.

(ii) If $v=106$ and $t=3$, find u and s.

(iii) If $v=76$ and $u=12$, find t and s.

(iv) A stone is thrown downwards with a velocity of 5 feet per second; how far must it fall to acquire a velocity of 133 feet per second?

61. I spend z pence in buying a apples at x farthings each and b pears at y farthings each.

(i) What formula connects a, b, x, y, z?

(ii) If $b=17$, $y=2$, $z=10$, what is the value of ax?

(iii) Find all the possible prices of an apple, if ax has this value.

62. A walks x miles an hour, B walks y miles an hour. They start at 1 p.m. from two places 24 miles apart and, walking towards one another, meet in t hours.
  (i) What is the formula connecting x, y and t?
  (ii) If $x=5$ and $y=3$, at what hour do they meet?

14. Subtraction. The equation $x+3=7$ is the algebraical statement of the question, "To what number must 3 be added that the sum may be 7?" To answer this question is "to subtract 3 from 7": the answer is written symbolically $7-3$.

Here, 7 is called the **minuend**, 3 the **subtrahend** and 4 the **remainder**.

In order to find the number $7-3$, start with the number 7 on the natural scale and count three backward (6, 5, 4). The last number used in this process of counting is 4, and we write $7-3=4$

If we start with 7 and count three backward and then three forward, or if we count three forward and then three backward, in each case the result is 7. This is expressed symbolically as follows:

$$7-3+3=7$$
$$\text{and} \quad 7+3-3=7.$$

Thus, to subtract 3 from any number and then to add 3 to the remainder, or to add 3 to the number and then to subtract 3 from the sum, is to leave the number unaltered. On this account, addition and subtraction are called **inverse operations.**

Def. The expression $a-b$ denotes the number to which, if b is added, the result is a. In other words, $a-b$ is defined by the equation

$$\mathbf{(a-b)+b=a.}$$

At present, the expression *$a-b$ is meaningless unless a is greater than b.*

When a series of operations, which are either additions or subtractions, is to be performed, it is at present assumed that *the operations are conducted in order from left to right*; thus, the expression $a-b-c$ denotes the number obtained by subtracting b from a, and then subtracting c from the remainder.

Ex. *Prove by counting that $9-3-2=9-2-3$.*

15. Division. The equation $3x=6$ is the algebraical statement of the question, "What is the number which, if multiplied by 3, produces 6?" To answer this question is "to divide 6 by 3," and the answer is written symbolically in any of the forms $6 \div 3$, $\frac{6}{3}$, $6/3$.

From the multiplication table, it appears that $6 \div 3 = 2$. Here, 6 is called the **dividend**, 3 the **divisor** and 2 the **quotient**.

The expression $7 \div 3$ *is at present meaningless*, for no number (that is to say no *whole* number) can be found which, if multiplied by 3, produces 7.

If a and b stand for two natural numbers, and if a natural number x can be found such that $x \times b = a$, then a is said to be (**exactly**) **divisible** by b, or to be a **multiple** of b.

Def. The expressions $a \div b$, $\frac{a}{b}$, a/b all denote the number which, if multiplied by b, produces a.

When a series of operations, which are either multiplications or divisions, is to be performed, it is at present assumed that *the operations are conducted in order from left to right*; thus, the expression $a \div b \div c$ stands for the number obtained by dividing a by b, and then dividing the quotient by c.

It will be proved later that multiplication and division are **inverse operations**.

Ex. *Verify that* $24 \div 2 \div 3 = 24 \div 3 \div 2$.

16. Use of Brackets. In the following cases, the bracket has the meaning assigned to it in Art. 9.

(1) The number $a-(b+c)$ is obtained by subtracting the sum of b and c from a.

(2) $(a+b) \div (c+d)$ stands for the quotient obtained by dividing the sum of a and b by the sum of c and d, and means exactly the same as

$$\frac{a+b}{c+d} \text{ or } (a+b)/(c+d).$$

(3) $a - bc + b \div c$ means the same as $a-(bc)+(b \div c)$. In finding the value of such an expression as this, the multiplications and divisions must be performed first (as indicated by the brackets), and then the additions and subtractions.

In the algebraical expression of number one bracket is often placed within another; in such a case it is necessary to make the brackets of different shapes. The forms commonly used are ( ), { }, [ ].

Thus $\{a-(b-c)\}^2$ means the square of the number denoted by $a-(b-c)$.

17. Terms of an Expression. In the expression

$$2a-3b-4c,$$

although $-3b$ and $-4c$, standing by themselves, have no meaning at present, yet *as forming parts of the expression* they have a meaning, and it is convenient to call $2a$, $-3b$ and $-4c$ **terms** of the expression $2a-3b-4c$.

EXERCISE V. MENTAL 1–27.

State the values of

1. $9-4-2$. **2.** $9-(4+2)$. **3.** $9-4+2$. **4.** $9-(4-2)$.

5. $9-4.2$. **6.** $(9-4)2$. **7.** $9-4\div 2$.

When $x=1$, state the values of

8. x^2+x-1. **9.** $10x^2-3x-2$. **10.** $3x^2-(x+1)$.

If $x=3$, state the values of

11. $10-2x$. **12.** $10-x^2$. **13.** $10-(12-x^2)$.

14. $20-x^2-(10-x^2)$. **15.** $(x-1)^2-(x-2)^2$.

If $a=5$ and $b=2$, state the values of

16. $2a\div b$. **17.** $a^3\div a^2$. **18.** $(a\div b)b$. **19.** $(ab)\div b$.

If $a=24$, $b=6$, $c=2$, state the values of

20. $a\div b\times c$. **21.** $a\div(b\times c)$. **22.** $(a\div b)\times c$. **23.** $\frac{a}{c}+b$.

24. $\frac{a+b}{c}$. **25.** $a+\frac{b}{c}$. **26.** $\frac{a}{b+c}$. **27.** $\frac{a}{b}+\frac{a}{c}$.

---

28. Describe the operations indicated in Examples 20-27, using words instead of signs.

29. Express with signs the following operations, arranging the results in the order described in the preceding definitions; use $\div$ to denote division:

(i) Multiply a by the quotient of b divided by c. (ii) Multiply by a the remainder when b is subtracted from c. (iii) Divide a by b and divide the result by c. (iv) Divide a by the product of c by b. (v) Divide a by the result of adding b to c.

30. By substituting 24 for c, 6 for b and 2 for a, verify that

$$c \times (a+b) = ca + cb,$$

but that $c \div (a+b)$ is not equal to $c \div a + c \div b$.

By substituting 7 for x, 2 for y and 1 for z, verify the identities

31. $x-(y+z)=x-y-z$.

32. $x-(y-z)=x-y+z$.

33. $x-2(y+z)=x-2y-2z$.

34. $x-2(y-z)=x-2y+2z$.

35. $(x+y)^2=x^2+2xy+y^2$.

36. $(x-y)^2=x^2-2xy+y^2$.

37. $x^2-y^2=(x+y)(x-y)$.

38. $(x+y)^2-(x-y)^2=4xy$.

39. $(x+y+z)^2=(x+y)^2+2(x+y)z+z^2$.

40. $(x+y-z)^2=x^2+y^2+z^2+2xy-2xz-2yz$.

41. $\{x+(y-z)\}\{x-(y-z)\}=x^2-(y-z)^2$.

42. $\{3x+2(y-z)\}\{3x-2(y-z)\}=9x^2-4(y-z)^2$.

43. In 20 years I shall be x years old; what is my present age?

44. What is A's present age if in x years he will be 40 years old?

45. What is A's present age if in x years he will be y years old?

46. From a stick a yard long five lengths each of x inches are cut and a length of 11 inches remains. What is x?

47. I have to make a journey of 100 miles; after travelling for a hours at x miles an hour, how far am I from my destination? If I still have 58 miles to go and $a=3$, how fast did I travel?

48. Starting with x pounds and y shillings, I spend z pence. Express the remaining sum in pence.

49. A runs at the rate of u yards per second, B runs at the rate of v yards per second. A gives B a start of x yards and catches him in t seconds. Express x in terms of u, v, t.

If $u=10$, $v=8$ and $t=20$, how much start does A give B?

50. A bullet is fired vertically upwards with a velocity of u feet per second: at the end of t seconds its velocity is v feet per second and it has risen s feet. It is found that u, v, t, s are connected by the formulae

$$v=u-32t, \qquad s=ut-16t^2.$$

(i) If $u=3000$ and $t=10$, find v and s.
(ii) If $u=2000$ and $v=16$, find t and s.
(iii) If $v=5$ and $t=71$, find u and s.
(iv) A shot is fired vertically upwards with a velocity of 3000 feet per second; after how many seconds will its velocity have been reduced to 280 feet per second? How far does it rise during this interval?

51. x and y are natural numbers connected by the equation $y=4x-21$.
(i) What are the values of y when x has the values 8, 9, 10 in succession?
(ii) What are the values of x when y has the values 31, 35, 39 in succession?
(iii) What is the least possible value of x?
(iv) What is the least possible value of y?

52. A and B walk at the rates of 3 and 4 miles an hour respectively. A walks for x hours and B for y hours.
(i) How much further has A gone than B?
(ii) If A has walked 5 miles further than B, what equation connects x and y?
(iii) From the equation obtained in (ii) find the values of x when y has in succession the values 7, 10, 13.
(iv) Find the values of y when x has successively the values 23, 27, 31.
(v) If in the equation obtained in (ii) x and y stand for natural numbers, what is the least possible value of x?

53. In a race, A gives B a start of a yards and catches him in t seconds.

If A runs x feet per second and B runs y feet per second,
(i) find a formula connecting x, y, t and a.
(ii) If $a=20$, $x=19$ and $t=15$, find the value of y.

18. The Laws of Subtraction. In such expressions as $a-b+c-d$, where a series of operations, which may be either additions or subtractions, is to be performed, it will be assumed that

(1) the operations are performed in order, from left to right, unless it has been shown that the result is not affected by the order of the operations;

(2) that the letters have such values that all the operations indicated are possible.

Ex. 1. *Explain why an impossible order of operations is indicated in* $2-5+6$.

Here we are directed to begin by subtracting 5 from 2, which at present is impossible.

Ex. 2. **Prove that $9-4-3+2=9-3+2-4$.**

The number denoted by $9-4-3+2$ is the last number used in the following process:

Start with the number 9 on the natural scale and count 4 numbers backward, 3 backward and 2 forward.

The number denoted by $9-3+2-4$ is the last number used in the following process:

Start with the number 9 and count 3 numbers backward, 2 forward and 4 backward.

In each case, the same number of symbols is counted, whether forward or backward, and the same number (4) is finally reached.

$$\therefore\ 9-4-3+2=9-3+2-4.$$

Experiments in counting like that described in the last example justify the following extension of the Law of Counting:

In counting backward or forward along the scale, the number finally reached is independent of the order of counting.

On this fact depends the truth of the **Commutative Law for Additions and Subtractions**, which is as follows:

In performing a series of operations, which may be either additions or subtractions, the result is not affected by the order of the operations, provided that these are conducted in an order which is possible.

19. Associative Law for Additions and Subtractions. This law asserts that

"**If a, b, c stand for any numbers, then**

$$\mathbf{a+(b+c)=a+b+c,}$$
$$\mathbf{a-(b+c)=a-b-c,}$$
$$\mathbf{a+(b-c)=a+b-c,}$$
$$\mathbf{a-(b-c)=a-b+c.}"$$

(Here it is assumed that a, b, c have such values that all the operations indicated are possible.)

These identities depend on the fact that *Number is independent of the order of counting.* We will prove the last, and leave the others as an exercise for the student.

Ex. 1. *Using natural numbers only,* (1) *explain the circumstances under which it is possible to prove the identity*

$$a-(b-c)=a-b+c.$$

(2) *Prove this identity, when it is possible to do so.*

Proof. (1) To find the number denoted by $a-(b-c)$, we are to subtract $(b-c)$ from a. Now $(b-c)$ is meaningless unless $b>c$; moreover $a-(b-c)$ is meaningless unless $a>(b-c)$.

Again, to find the number $a-b+c$, we are to begin by subtracting b from a: this is impossible unless $a>b$; if this is the case, it follows that $a>(b-c)$. Hence the necessary and sufficient conditions are $a>b>c$.

(2) The number denoted by $a-(b-c)$ is the last number used in the following process: Start with the number a on the scale and count $(b-c)$ numbers backward. This process is the same as the following: Start with the number a, count b backward and then c forward. The last number is therefore also denoted by $a-b+c$;

$$\therefore\ a-(b-c)=a-b+c.$$

Ex. 2. *Simplify* $(5x-4y)-(3x-2y)$.

$$\begin{aligned}(5x-4y)-(3x-2y)&=5x-4y-3x+2y && (\textit{Associative Law})\\ &=5x-3x-4y+2y && (\textit{Commutative Law})\\ &=(5x-3x)-(4y-2y) && (\textit{Associative Law})\\ &=2x-2y.\end{aligned}$$

EXERCISE VI. MENTAL.

NOTE. *Letters, the values of which are not given, are supposed to have such values that the operations indicated are possible.*

Simplify

1. $5a-4a+3a-2a$.
2. $2x+3y-(x+3y)$.
3. $2x-3y-(x-3y)$.
4. $97a-(45a+50a)$.
5. $98a-46a-50a$.
6. $a^2-ab-(ab-b^2)$.
7. $5a-2b-(a-b)$.
8. $a^2-ab-(b^2+ab)$.
9. $6a-3b-(2b+3a)$.

What are the following numbers on the natural scale:

10. The 5th number before a?
11. The ath number before $a+b$?
12. The bth number after $a-2b$?
13. The number just before $n-3$?
14. The number just after $n-3$?
15. The nth number before $n+2$?
16. The nth number after $n-2$?

What is the number

17. Greater than $n-5$ by 7?
18. Less than $n-5$ by 7?
19. Greater than $a-b$ by b?
20. Less than $a-b$ by b?
21. Greater than a by $a-b$?
22. Less than a by $a-b$?
23. Greater than $3x+y$ by $x-y$?
24. Less than $3x+y$ by $x-y$?

25. A balloon rises 500 feet, then rises 400 feet more, then sinks 800 feet; if the balloon always starts from the ground, name any other order in which these events could happen.

26. You have 10 pence to spend and proceed as follows: (i) earn $6d.$, (ii) spend $12d.$, (iii) spend $3d.$ If you always start with $10d.$, there are several orders in which these operations cannot be conducted. Name them, and say how much you must borrow, in order that they may be possible in any order.

27. In doing a journey of 100 miles, I travel for a hours by train at x miles an hour, for b hours by boat at y miles an hour and walk the rest of the way. How far do I walk?

28. From a rod a feet long, a length of $(b-c)$ feet is cut off. How long is the remaining part of the rod? Give the answer in two different forms.

29. I have 5 shillings in my purse and buy a apples at x pence each and b pears at y pence each. How many pence have I left?

30. State three consecutive numbers of which (i) x is the middle one, (ii) y is the greatest.

31. State three consecutive odd numbers of which (i) $2n-1$ is the middle one, (ii) $2n-3$ is the greatest.

32. What is the sum of three consecutive even numbers of which $2n$ is the middle one?

33. What is the sum of three consecutive numbers of which x is the greatest?

34. What is the sum of three consecutive odd numbers of which x is the greatest?

35. Find the sum of

$$3a^2-5a+2b,\ 7a-4b-3b^2,\ 2a^2-b+a-2b^2.$$

36. Subtract $2b-(c-a)$ from $2c-(a-b)$.

37. What expression must be subtracted from $5x^3+2x-3$ to leave $3x^3-2x^2+7$?

38. Find the sum of $2a-(5c+3b)$ and $4b-(a-3c)$, and subtract $a-3c$ from the sum.

39. Add together

$$2x+3y-4z,\ 5z+5x-4y,\ \text{and}\ 7y-3x-8z.$$

From the sum of these subtract $2x-y-6z$.

40. Find the sum of

$$3x^3+(5x^2y-xy^2),\ 3xy^2-(6x^2y-4y^3),\ 2x^2y-(xy^2+y^3).$$

Subtract $3x^2y-2xy^2$ from the result.

CHAPTER III.

EQUATIONS AND PROBLEMS.

20. Rules for Equalities. Consider the following questions:

(1) What is the result of adding one number to another?

(2) What is the result of multiplying one number by another?

To each of these questions there is always one answer and only one, and in each case the answer is a definite number.

Next, consider the questions:

(3) What is the result of subtracting one number from another?

(4) What is the result of dividing one number by another?

For either of these questions, it may happen that no number exists which is an answer: if such a number exists, then there is one and only one such number.

In order that it may be always possible to answer the questions (3) *and* (4), *two new kinds of numbers are invented. These will be considered later.*

All processes in Algebra depend on

(1) The laws which govern the fundamental operations.

(2) Certain theorems called "Rules for Equalities and Inequalities."

Rules for Equalities are as follows:

(1) *If* $\mathbf{a=b}$, *then* $\mathbf{b=a}$.

For either of these equations means that a and b stand for the same number.

If $\mathbf{a=b}$, *then*

(2) $\mathbf{a+x=b+x}$; (3) $\mathbf{a-x=b-x}$;

(4) $\mathbf{a\times x=b\times x}$; (5) $\mathbf{a\div x=b\div x}$.

Proof of (2). a and b stand for the same number; and there is one and only one answer to the question, "What is the result of adding one number to another number?"

(6) *If* $\mathbf{a=b}$ *and* $\mathbf{b=c}$, *then* $\mathbf{a=c}$.

For a, b, c stand for the same number.

If $\mathbf{a=b}$ *and* $\mathbf{x=y}$, *then*

(7) $\mathbf{a+x=b+y}$; (8) $\mathbf{a-x=b-y}$;

(9) $\mathbf{a\times x=b\times y}$; (10) $\mathbf{a\div x=b\div y}$.

Proof of (8). *If any answer exists* to the question, "What is the result of subtracting one number from another number"? then there is only one answer; now a and b stand for the same number, also x and y stand for the same number,

$$\therefore\ a-x=b-y.$$

21. Converse Theorems. Referring to the theorems of Art. 20, in (1) the given condition is $a=b$; this is called the **hypothesis**. The **conclusion** is $b=a$. *One theorem is called the* **converse** *of another when the hypothesis of each of the theorems is the conclusion of the other. If a theorem is true, it does not necessarily follow that the converse of the theorem is true.*

The *converse* of each of the theorems (1–5) is true. We will prove the converse of (2), which is as follows:

If $a+x=b+x$, *then* $a=b$.

Proof. $a+x=b+x$; (*Hypothesis*)

$\therefore\ (a+x)-x=(b+x)-x$; (*Third Rule for Equalities*)

$\therefore\ a+x-x=b+x-x$; (*Associative Law*)

$\therefore\ a=b$.

Consider the following theorems:

(i) If $a=b$ and $x=y$, then $a+x=b+y$;
(ii) If $a+x=b+y$ and $a=b$, then $x=y$;
(iii) If $a+x=b+y$ and $x=y$, then $a=b$.

The first of these is Theorem 7 of Art. 20, the second and third follow from Theorem 8 of that Article. The three theorems show that, if any *two* of the statements $a=b$, $x=y$, $a+x=b+y$ are true, the *third* is also true.

When three theorems are related in this way, *any two of them are often called converses of the third.*

22. Rules for Inequalities.

Rules for Inequalities are as follows:

(1) If $\mathbf{a>b}$ and $\mathbf{b>c}$, then $\mathbf{a>c}$.
(2) If $\mathbf{a<b}$ and $\mathbf{b<c}$, then $\mathbf{a<c}$.

If $\mathbf{a>b}$, then

(3) $\mathbf{a+x>b+x}$; (4) $\mathbf{a-x>b-x}$;
(5) $\mathbf{ax>bx}$; (6) $\mathbf{a\div x>b\div x}$.

If $\mathbf{a<b}$, then

(7) $\mathbf{a+x<b+x}$; (8) $\mathbf{a-x<b-x}$;
(9) $\mathbf{ax<bx}$; (10) $\mathbf{a\div x<b\div x}$.

If $\mathbf{a>b}$ and $\mathbf{x>y}$, then

(11) $\mathbf{a+x>b+y}$; (12) $\mathbf{ax>by}$.

If $\mathbf{a<b}$ and $\mathbf{x<y}$ then

(13) $\mathbf{a+x<b+y}$; (14) $\mathbf{ax<by}$.

Proofs of (4) and (12) will be given and the rest will be left as an exercise.

Proof of (4). If $a>b$, then, by definition, a follows b on the natural scale (Art. 4).

$\therefore$ $a-x$ follows $b-x$ on the scale; $\therefore$ $a-x>b-x$.

Proof of (12). If $a>b$ and $x>y$, then

$ax>bx$ and $bx>by$; (*Rule* 5)

$\therefore$ $ax>by$. (*Rule* 1)

23. Equations and Problems. The equation

$$2x+3=17$$

is equivalent to the question, "If $2x+3$ stands for 17, what number does x stand for?" In seeking an answer to this question, *we begin by assuming that an answer exists and that x stands for the required number.* We proceed as follows:

$2x+3$ stands for 17. ....(α)

Hence if 3 is subtracted from $2x+3$ and from 17, the same number will result,

$\therefore$ $2x$ stands for 14; ....(β)

and if $2x$ and 14 are divided by 2, the same number will result,

$\therefore$ x stands for 7. ....(γ)

So far it has been shown that if there is an answer to the original question, the answer is 7. In order to show that 7 is an answer, we start with the assumption that

x stands for 7. ... .(γ)

Multiplying x and 7 by 2, the same number must result,

$2x$ stands for 14. ....(β)

If 3 is added to $2x$ and to 14, the same number results,

$2x+3$ stands for 17. ....(α)

Thus **all the steps in the process are reversible**, and we see that 7 is an answer and is the only answer to the question.

This value 7 of x, which makes both sides of the given equation equal, is said to satisfy the equation, and is called the solution of the equation. The process of finding the solution of an equation or of proving that the equation has no solution is called **solving the equation.**

The process in this article can be written more concisely thus: The given equation is

$2x+3=17$. ....(α)

Subtracting 3 from each side,

$2x=14$. ....(β)

Dividing each side by 2, $x=7$. ....(γ)

Since each step in this process is reversible, the solution is 7.

24. Rules for Equations. The considerations of the last article lead to the following rules:

An equation is not *altered*—that is to say, it continues to be satisfied by the same value (or values) of the unknown

(1) If the sides of the equation are interchanged—so that the right-hand side becomes the left-hand side.

(2) If the same number is added to each side.

(3) If the same number is subtracted from each side.

(4) If each side is multiplied by the same given number.

(5) If each side is divided by the same given number.

Here "given number" means a number whose value is *known*. It will be shown later that to multiply or to divide each side of an equation by an expression containing the *unknown* is, in general, to alter the equation.

Ex. *Divide £80 between A, B and C so that A gets three times as much as B and C gets £10 more than B. Verify your result.*

Suppose that B gets £x, then A gets £$3x$ and C gets £$(x+10)$.

Thus we have: $3x+x+(x+10)=80$,

$$\therefore\ 3x+x+x+10=80,$$

$$\therefore\ 5x+10=80.$$

Subtracting 10 from each side,

$$\therefore\ 5x=70.$$

Dividing each side by 5,

$$\therefore\ x=14.$$

Thus, A's share = £$3x$ = £(3.14) = £42,

B's share = £x = £14,

C's share = £$(x+10)$ = £24.

In order to verify the result, it must be shown that the conditions of the problem are satisfied. These are as follows:

(1) Money divided between A, B, C = £$(42+14+24)$ = £80.

(2) A's share = £42 = £(3.14) = three times B's share.

(3) C's share = £10 more than B's share.

EXERCISE VII. MENTAL 1–21.

1. You are told that A is as tall as B and that B is as tall as C. What can you say about the heights of A and C?
2. If $x=y$ and $x=z$, what can you say about y and z?
3. If $x+3=2y+4$, explain how to derive from this the equation $x=2y+1$. If $y=3$, what is the value of x?
4. If $2x=6$, quote the rule which enables you to say that $x=3$.
5. If $2x+5=11$, what number does x stand for?
6. The equation in question 5 contains the solution of this problem, "If 5 is added to ......" Complete this statement.
7. What value of x satisfies the equation $3x+4=19$? Verify your answer.
8. If $2(x+3)=12$, what is the value of $x+3$? What is the value of x?
9. The equation in question 8 contains the solution of this problem, "If 3 is added to a certain number ......" Complete this statement.

Solve the following equations, and in each case verify the solution:

10. $2x+9=41$.
11. $7+5x=17$.
12. $29=9+2x$.
13. $31=4+9x$.
14. $6x-5=37$.
15. $6(x-5)=36$.
16. $3+2(x+2)=13$.
17. $(3+2x)+2=13$.
18. $3+2(x-2)=13$.

19. The steps in the solution of the equation $x+(x+1)+(x+2)=93$ are as follows:

$$x+(x+1)+(x+2)=93,$$
$$\therefore\ x+x+1+x+2=93, \quad \ldots\ldots\ldots\ldots(1)$$
$$\therefore\ x+x+x+1+2=93, \quad \ldots\ldots\ldots\ldots(2)$$
$$\therefore\ 3x+3=93,$$
$$\therefore\ 3x=90, \quad \ldots\ldots\ldots\ldots(3)$$
$$\therefore\ x=30. \quad \ldots\ldots\ldots\ldots(4)$$

State the laws or rules in virtue of which the numbered steps are permissible.

20. The equation in question 19 contains the solution of the problem, "The sum of three ...... numbers ......". Complete the statement of the problem.

21. If I write down all the numbers from 1 to 99 thus: 1, 2, 3, 4, ..., 99:

(i) How many numbers of this series are there before 39?

(ii) How many after 39?

If x is any one of the numbers between 1 and 99

(iii) How many numbers are there before x?

(iv) How many after x?

(v) If x is the middle number of the series, the equation to find x is $x-1=99-x$. Explain this.

---

22. The first two steps in solving the equation $x-1=99-x$ are

$$x+x-1=99, \qquad\text{(1)}$$

$$x+x=99+1. \qquad\text{(2)}$$

What is done to each side of the equation in these steps?

23. Which is the middle number in the series 1, 2, 3, ..., 155? Explain how to verify your answer.

24. There are two middle numbers in the series 1, 2, 3, 4, 5, 6, 7, 8. Name them.

25. If $x-1$ and x are the two middle numbers of the series 1, 2, 3, 4, ..., 200, the equation to find x is $x-1-1=200-x$. Explain this.

26. If $2x-2=200$, what is the value of x? What are the two middle numbers of the series 1, 2, 3, 4, ..., 200?

27. What is the sum of 5 consecutive numbers of which x is the middle number?

28. A man's salary rises £x every year: his salary for his third year of service is £y.

(i) What is his salary for the first year?

(ii) What is it for the fifth year?

(iii) How much does he earn altogether during his first five years of service?

29. x and y stand for two numbers such that $5x+y=47$ and $2x+y=23$. Find (i) the value of $(5x+y)-(2x+y)$, and (ii) the values of x and y.

30. x and y stand for two numbers such that $3x+2y=47$ and $5x-2y=41$. Find (i) the value of $(3x+2y)+(5x-2y)$, and (ii) the values of x and y.

31. x, y and z stand for three numbers such that $3x+(y+z)=43$ and $5x-(y+z)=45$.
   (i) Find the values of x and $y+z$.
   (ii) If it is also given that $y=4z$, find the values of y and z.

32. x and y are two numbers such that $3(x+y)+4(x-y)=32$ and $3(x+y)-4(x-y)=16$.
   (i) Find the values of $(x+y)$ and $(x-y)$.
   (ii) Find the values of x and y.

33. x and y are two natural numbers whose product is 6.
   (i) Write down all the possible values of x and the corresponding values of y.
   (ii) If it is also given that $x+y=5$, what are the possible values of x and the corresponding values of y?

34. x and y stand for two numbers such that

$$xy+x=21 \quad \ldots\ldots\ldots\ldots(1)$$
$$\text{and} \quad 10x-xy=56. \quad \ldots\ldots\ldots\ldots(2)$$

   (i) Write down the equation obtained by substituting the value of xy, obtained from equation (1) in equation (2).
   (ii) Find the values of x and y.

35. The length of a room is l feet, its breadth b feet, and its perimeter p feet. (i) What is the formula connecting p, l, b? (ii) If $p=98$, $l=27$, what is b? (iii) If $p=86$, $b=19$, what is l? (iv) If $l=30$, $b=20$, what is p?

36. The sum of two numbers is 93 and the greater exceeds the less by 39. Find the numbers. (Let x stand for the smaller number.)

37. The sum of two numbers is 103 and their difference is 27. Find the numbers. (Let x stand for the smaller number.)

38. The difference between two numbers is 29, and if the greater is added to 5 times the less, the sum is 83. Find the numbers.

39. Divide £219 between A and B so that A gets £47 more than B.

40. Divide £100 between A and B so that A's share exceeds 3 times B's share by £16.

41. A man is three times as old as his son. The son's age is now n years. What will be the sum of their ages in a years? If the sum of their ages in a years is $10a$, find n in terms of a.

42. The sum of two consecutive numbers is 97 ; find them.

43. Find, in its simplest form, the sum of three consecutive odd numbers the least of which is greater than $2n$ by unity.

44. The sum of three consecutive odd numbers is 333 ; find them.

45. (i) If the sum of the numbers x, $2x$, $3x$ is 534, find x.

(ii) Divide £2. 4*s.* 6*d.* between A, B and C so that B gets twice as much as C and A gets three times as much as C.

46. The earnings in shillings of a boy, a woman and a man are x, $2x$ and $3x$ respectively.

(i) What are the total earnings of 3 men, 4 women and 5 boys?

(ii) If the total earnings of these people amount to £11, what does a man earn?

47. A has twice as much money as B, and C has £6 less than B; if B has £x,

(i) How much have A and C respectively?

(ii) How much have A, B and C together?

(iii) If A, B and C have £54 between them, how much has B?

48. Find two consecutive numbers such that 3 times the greater added to twice the less amounts to 183.

49. One number exceeds another number by 10, and 4 times the greater added to 5 times the less amounts to 139. Find the numbers.

50. A sum of money consists of x sovereigns, $(x+10)$ shillings and $3x$ pence, and the value of the sum is £11. 2*s.* 6*d.* Find the value of x.

51. A runs 18 feet per second and B runs 15 feet per second. A gives B a start of x seconds, and when A has run for y seconds he catches B. Find y in terms of x.

CHAPTER IV.

LAWS OF MULTIPLICATION.

25. Distributive Law. The fundamental laws of **multiplication** are the **Distributive**, the **Commutative** and the **Associative** Laws, and on these all processes of multiplication depend. The Distributive Law asserts that if a, b, c stand for any numbers, then

$$\mathbf{(a+b)\,c=ac+bc}$$

and $$\mathbf{c\,(a+b)=ca+cb.}$$

This has been proved in Art. 12.

It has been explained that abc, $a\,.\,b\,.\,c$, $a \times b \times c$ all stand for the number obtained by multiplying a by b and then multiplying the result by c. Also, from the general meaning of a bracket, it follows that $a(bc)$ denotes the number obtained by multiplying a by the product of b and c, and that $(ab)c$ means the same as abc.

26. Commutative and Associative Laws for Multiplication. These laws are as follows:

The Commutative Law: A product is independent of the order of its factors.

The Associative Law: The product of any collection of numbers may be found by associating the numbers in groups, finding the product of the numbers in each group and then finding the product of the results.

To establish these laws generally, both laws must be considered together. It has been shown that $ab=ba$; this is an instance of the Commutative Law. Consider the following examples:

Ex. 1. **Prove that $4.3.2=3.2.4$.**

Proof

$$\begin{aligned} 4.3.2 &= 4.3+4.3 && (\textit{Def. of Multiplication}) \\ &= 3.4+3.4 && (\because ab=ba) \\ &= (3+3)4 && (\textit{Distributive Law}) \\ &= (3.2)4 && (\textit{Def. of Multiplication}) \\ &= 3.2.4. && (\textit{Def.}) \end{aligned}$$

Ex. 2. **Prove that $4.3.2=4.(3.2)$.**

Proof

$$\begin{aligned} 4.3.2 &= (3.2).4 && (\textit{Ex. } 1) \\ &= 4.(3.2). && (\because ab=ba) \end{aligned}$$

It has thus been shown that to multiply 4 by 3 and then to multiply the result by 2 is to multiply 4 by the product of 3 and 2.

In Ex. 1, 2, *the argument is general* and holds if 4, 3, 2 are replaced by any numbers a, b, c;

$$\therefore\ abc=bca \text{ and } bca=cab;$$

$$\text{also } abc=a(bc)=a(cb)=acb.$$

hence **the expressions** abc, acb, bca, bac, cab, cba **all denote the same number.**

The Commutative and Associative Laws can now be established for products of 4, 5, 6, ... factors in succession.

Ex. 3. **Prove that $abcd=a(bcd)$.**

Proof

$$\begin{aligned} abcd &= a(bc)d && (\because abc=a(bc)) \\ &= (bc)da && (\because xyz=yzx) \\ &= (bcd)a && (\textit{Def.}) \\ &= a(bcd). && (\because mn=nm) \end{aligned}$$

Ex. 4. **Prove that $abcd=bdac$.**

Proof

$$\begin{aligned} abcd &= bacd && (\because ab=ba) \\ &= b(acd) && (\textit{Ex. } 3) \\ &= b(dac) && (\because acd=dac) \\ &= bdac. && (\textit{Ex. } 3) \end{aligned}$$

27. Powers of Numbers. It has been explained that a^2 stands for $a \times a$, and that a^3 stands for $a \times a \times a$. In the same way $a \times a \times a \times a$ is called the **fourth power of** a, or a **to the fourth (power)**, and is written $\mathbf{a^4}$.

Def. The continued product of n equal numbers each represented by a is called the nth **power of** a, or "a to the nth," and is written $\mathbf{a^n}$.

The process of finding a^n is called "raising a to the nth power," and the operation of raising a number to any given power is called **involution**.

In the expressions a^2, a^3, a^4, ... a^n ..., the numbers 2, 3, 4, ... n ... are called **indices**, or **exponents**, as indicating the particular power to which the number a is raised.

In this connection a is called the first power of a, and may be written a^1.

Ex. 1. *Prove that* $a^2 . a^3 = a^{2+3}$.

$$\begin{aligned} a^2 . a^3 &= (a . a) . (a . a . a) && (Def.) \\ &= a . a . a . a . a && (Associative\ Law) \\ &= a^{2+3}. && (Def.) \end{aligned}$$

Ex. 2. *Prove that* $(a^2)^3 = a^{2\times 3}$.

$$\begin{aligned} (a^2)^3 &= a^2 . a^2 . a^2 && (Def.) \\ &= (a . a) . (a . a) . (a . a) && (Def.) \\ &= a . a . a . a . a . a && (Associative\ Law) \\ &= a^{2\times 3}. && (Def.) \end{aligned}$$

Ex. 3. *Prove that* $(ab)^2 = a^2b^2$.

$$\begin{aligned} (ab)^2 &= (ab)(ab) && (Def.) \\ &= abab && (Associative\ Law) \\ &= aabb && (Commutative\ Law) \\ &= (aa)(bb) && (Associative\ Law) \\ &= a^2b^2. && (Def.) \end{aligned}$$

The argument in the last three examples *is general* and does not depend on the particular numbers chosen as indices. We

are therefore able to state the following fundamental **index laws**:

If a, b, m, n stand for any numbers, then

$$(1)\ \mathbf{a^m . a^n = a^{m+n}},\qquad (2)\ \mathbf{(a^m)^n = a^{mn}},$$

$$(3)\ \mathbf{(ab)^m = a^m b^m}.$$

Ex. 4. *Multiply* $5x^2y$ *by* $7x^4y^3$.

$$\begin{aligned}(5x^2y)(7x^4y^3) &= 5 . x^2 . y . 7 . x^4 . y^3 && (\textit{Associative Law})\\ &= 5 . 7 . x^2 . x^4 . y . y^3 && (\textit{Commutative Law})\\ &= (5 . 7)(x^2x^4)(yy^3) && (\textit{Associative Law})\\ &= 35x^6y^4. && (\textit{First Index Law})\end{aligned}$$

Ex. 5. *Simplify* $(3x^3)^3$.

$$\begin{aligned}(3x^3)^3 &= 3^3(x^3)^3 && (\textit{Third Index Law})\\ &= 3^3 . x^{3\times 3} && (\textit{Second Index Law})\\ &= 27x^9.\end{aligned}$$

If n and a stand for *any* two given numbers, there is always one answer and only one answer to the question, "What is the nth power of a?" We have therefore the following **Rule for Equalities**:

$$\textit{If}\ \mathbf{a = b},\ \textit{then}\ \mathbf{a^n = b^n}.$$

28. Substitution. The employment of the method of substitution is that which gives to the processes of Algebra their great power as compared with the processes of Arithmetic.

For example, it has been shown that if a, b, c stand for any three numbers, then $(a+b)c = ac+bc$; if then we substitute for a, b, c any three *expressions* denoting numbers, we shall obtain another identity.

Ex. 1. *From the identity* $(a+b)c = ac+bc$, *by substituting* $(2x^2)$ *for* a, $(3y^2)$ *for* b, *and* $(5xy)$ *for* c, *obtain the expanded form of the product* $(2x^2+3y^2)(5xy)$.

We have
$$\begin{aligned}(2x^2+3y^2)(5xy) &= (2x^2)(5xy) + (3y^2)(5xy)\\ &= 10x^3y + 15xy^3.\end{aligned}$$

Thus, to multiply $(2x^2+3y^2)$ by $(5xy)$, we multiply each of the expressions $(2x^2)$ and $(3y^2)$ by $(5xy)$ and add the results.

Ex. 2. *From the identity $c(a+b)=ca+cb$, by substituting $(2x)$ for a, $(3y)$ for b, and $(4x+5y)$ for c, obtain the expanded form of the product $(4x+5y)(2x+3y)$.*

We have
$$\begin{aligned}(4x+5y)(2x+3y)&=(4x+5y)(2x)+(4x+5y)(3y)\\&=8x^2+10xy+12xy+15y^2\\&=8x^2+22xy+15y^2.\end{aligned}$$

Thus, to multiply $(4x+5y)$ by $(2x+3y)$ we multiply $(4x+5y)$ by $2x$ and $(4x+5y)$ by $3y$ and then add the results.

EXERCISE VIII. MENTAL 1–44.

Simplify the following :

1. $xxx+xxx.$ **2.** $xx+xx+xx.$

3. $xyz+yzx+zxy.$ **4.** $(xyz)(yzx)(zxy).$

5. $(zx)(xy)+(xy)(yz)+(yz)(zx).$ **6.** $x^5\times x^7.$

7. $(5x^5)\times(7x^7).$ **8.** $(2ab).(3a^2b^3).$

9. $(5x^3).(6x^2y^2).$ **10.** $(abc)(b^2c^3).$ **11.** $x^2+x^2+x^2.$

12. $x^2.x^2.x^2.$ **13.** $(x^2)^3.$ **14.** $(x^3)^2.$

15. $(2x^2)^3.$ **16.** $(3x^3)^2.$ **17.** $(3x^3)^4.$

18. $(4x^4)^3.$ **19.** $(x^3)^2+(x^2)^3.$ **20.** $(x^4)^5.$

Remove the brackets in the following and simplify :

21. $3(2x+3y).$ **22.** $4a^2(2a+5b).$

23. $ab(a^2+b^2).$ **24.** $a^2b^2(a+b).$

25. $2x^2(3x^3+1).$ **26.** $3x^2y(4x^3y+3xy^3).$

27. $5x^2(2x^2+3x+4).$ **28.** $x(x+y)+y(x+y).$

Express the following in factors by inserting a bracket :

29. $x^2+x.$ **30.** $2x^3+2x^2.$ **31.** $3x^3+6x.$

32. $a^2b+ab^2.$ **33.** $x^2yz+xy^2z+xyz^2.$ **34.** $x^2y^2+xy.$

35. $10x^2y^3+5x^3y^2.$ **36.** $x(x+y)+y(x+y).$

37. $(x+y)^2+(x+y).$ **38.** $(x+1)^2+(x+1).$

39. $x(x+2y)+y(x+2y).$ **40.** $x(x+2y)+2y(x+2y).$

By arranging the factors in the most suitable order, find the values of

41. $2.2.2.5.5.$ **42.** $125.2^4.$ **43.** $2^4.5^3.$ **44.** $2^3.5^5.$ **45.** $25^3.4^4.$

Simplify the following :

46. $x(y+z)+y(z+x)+z(x+y)$. **47.** $x(2y+3z)+y(2z+3x)+z(2x+3y)$.

48. In the identity $c(a+b)=ca+cb$.

(i) Substitute $(x+y)$ for c and deduce the expanded form of $(x+y)(a+b)$.

(ii) Substitute $(a+b)$ for c and prove the identity

$$(a+b)^2=a^2+2ab+b^2.$$

(iii) Substitute $(a^2+2ab+b^2)$ for c and prove the identity

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

49. In the identity $(a+b)^2=a^2+2ab+b^2$.

(i) By substituting $4x$ for a and $5y$ for b, find the expanded form of $(4x+5y)^2$.

(ii) By substituting $2x^2$ for a and $3y^2$ for b, find the expanded form of $(2x^2+3y^2)^2$.

50. By making the necessary substitutions in the identity

$$(a+b)^2=a^2+2ab+b^2$$

find the squares of

(i) $x+1$. (ii) $x+3$. (iii) $x+5$. (iv) $2x+1$.
(v) $4x+1$. (vi) $2x+3$. (vii) $3x+4$. (viii) $5x+6y$.
(ix) $7x+8y$. (x) x^2+y^2. (xi) $3x^2+2y^2$. (xii) x^3+y^3.

51. By making the necessary substitutions in the identity

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

find the cubes of the following in their expanded forms :

(i) $x+2$. (ii) $3x+1$. (iii) $2x+5y$. (iv) x^2+y^2. (v) $xy+4z^2$.

52. Expand the following products :

(i) $(x+3)(x+4)$. (ii) $(x+5)(x+6)$. (iii) $(x+10)(x+12)$.
(iv) $(5x+1)(2x+1)$. (v) $(2x+3)(4x+5)$. (vi) $(5x+6)(7x+8)$.
(vii) $(x+1)(x^2+x+1)$. (viii) $(2x+3y)(2x^2+3y^2)$.

53. Obtain the expanded form of $(x+1)^4$ and verify the result by substituting 1 for x.

54. Simplify the following expressions and verify the results by substituting 1 for x.

(i) $(x+1)^2+2(x+1)(x+2)+(x+2)^2$.

(ii) $x^3+3x^2(x+2)+3x(x+2)^2+(x+2)^3$.

55. Explain a way of *writing down* the answers in Ex. 54 by making proper substitutions in formulae (ii) and (iii) of Ex. 48.

29. Dimension and Degree. Consider any term of the expression $4x^2y^3z+5z^2x^2+6x^3$. The term $4x^2y^3z$ written at full length is $4xxyyyz$, and it contains 6 factors represented by letters.

Each factor of a term, which is represented by a letter, is called a **dimension** of the term, and the number of these factors is called the **degree** of the term.

Thus the term $4x^2y^3z$ is said to be of the sixth degree; it is also said to be of six dimensions in x, y, z.

A term of six dimensions is said to be of **higher degree** than one of five dimensions.

The **degree of an expression** is that of its term of highest degree.

The first, second and third terms of $4x^2y^3z+5z^2x^2+6x^3$ are of the sixth, fourth and third degree respectively, and the expression itself is of the sixth degree.

An expression is said to be **homogeneous** when all its terms are of the same degree. Thus $x^3-2y^2z+4xyz$ is a homogeneous expression of the third degree in x, y, z.

It is often necessary to consider expressions in which certain letters (such as a, b, c, ...) always stand for the *same* known numbers, and other letters (such as x, y, z, ...) receive *different* values. In such a case a, b, c, ... are called **constants** and x, y, z ... are called **variables**. In determining the degree of an expression containing constants and variables, the constant letters are not counted; thus $ax^2+bxy+cy^2$ is homogeneous and of *the second degree in x and y*, and a, b, c are called the **coefficients** of x^2, xy, y^2 respectively.

If x is variable, an expression (like ax^2+bx+c) which is of the second degree in x is said to be **quadratic** in x.

30. Arithmetical G.C.M. A number which is a factor of two or more numbers is called a **common factor**, or a **common measure** of the numbers.

Numbers which have no common factor except unity are said to be **prime** to each other.

A number which has no factors except itself and unity is called a **prime number**.

The greatest number which is a factor of two or more numbers is called the **Greatest Common Measure** of the numbers and is denoted by the letters G.C.M.

Ex. 1. *Express each of the numbers 108 and 180 as a product of prime factors and find the* G.C.M. *of the given numbers.*

$$108 = 2^2.3^3; \quad 180 = 2^2.3^2.5.$$

Assuming that a number can be expressed as a product of prime factors in one way only, it appears that the highest power of 2 which is a factor common to 108 and 180 is 2^2, the highest power of 3 which is a common factor is 3^2, and the numbers have no other common factor except unity.

Hence the G.C.M. is $2^2.3^2$ or 36.

Ex. 2. *If a, b, c represent* prime *numbers, what is the* G.C.M. *of $a^3b^5c^7$ and $a^2b^4c^5$?*

The G.C.M. is $a^2b^4c^5$.

31. Algebraical H.C.F. The Highest Common Factor of two or more algebraical expressions containing letters x, y, z ... is the expression of highest degree in x, y, z ... which is a factor of the given expressions, and is denoted by the letters H.C.F.

Ex. *Find the* H.C.F. *of* (i) $x^5y^2z^2$, x^3yz^4, $x^2y^3z^5$.

(ii) $a^2(x+y)^3(x-y)^4$ *and* $a^3(x+y)^2(x-y)^3$.

(i) The H.C.F. is x^2yz^2. (ii) The H.C.F. is $a^2(x+y)^2(x-y)^3$.

It should be noticed that the H.C.F. of two algebraical expressions is not necessarily the G.C.M. of the numerical values of the expressions.

Thus, if $x=6$ and $y=4$,

$$x^2y = 6^2.4 = (6.4.2).3,$$
$$xy^2 = 6.4^2 = (6.4.2).2.$$

The G.C.M. of the numerical values of x^2y and xy^2 is therefore 6.4.2, whilst the H.C.F. of x^2y and xy^2 is xy or 6.4.

It will be seen that in accordance with the above definition, the H.C.F. of $6x^2y$ and $9xy^2$ may be taken to be *either* xy *or* $3xy$.

It is convenient that the H.C.F. of two or more algebraical expressions should contain as a factor the G.C.M. of any numerical factors which may occur in the given expressions.

Thus the H.C.F. of $6x^2y$ and $9xy^2$ is taken to be $3xy$.

32. Algebraical L.C.M. The Lowest Common Multiple of two or more algebraical expressions is the expression of least dimensions, which contains each of the given expressions as a factor and is denoted by the letters L.C.M.

Ex. *Find the* L.C.M. (i) *of* $6x^7yz^5$, $12x^5y^2z^2$ *and* $18x^3yz^4$.

(ii) *of* $x^3(x+y)^2(x-y)$ *and* $x^2y(x+y)(x-y)^3$.

(i) The L.C.M. is $36x^7y^2z^5$. (ii) The L.C.M. is $x^3y(x+y)^2(x-y)^3$.

EXERCISE IX. (MENTAL 1–21.)

State

1. All the factors of the second degree (i) of xyz; (ii) of x^2y.

2. All the factors of three dimensions of (i) x^2yz; (ii) of x^4y.

3. All the factors (disregarding unity) common to x^2yz and axy^2.

4. If a, b, c represent prime numbers, what is the G.C.M. of the numbers denoted by ab^2c^3, bc^2a^3, $c^4a^2b^3$?

Find the highest common factor of

5. $2ab$, $4ax$. **6.** $3axy$, $4aby$. **7.** a^2xy, ax^2z.

8. $9a^2b^2c^3$, $12a^3bc$. **9.** $12a^3c$, $12ac^3$. **10.** $a^{10}b^7c^3$, $a^3b^7c^{10}$.

11. $24x^3y^2z$, $30xy^3z^2$, $36x^2yz^3$. **12.** $apxy$, bp^2qy, cp^3ryz.

13. $6a(a+b)$ and $9(a+b)^2$. **14.** $a^2(a+2b)$ and $(a+2b)^2(a-2b)$.

15. $3a+3b$ and $2a+2b$.

Find the lowest common multiple of

16. ab, bc. **17.** $3axy$, $4aby$. **18.** a^2xy, b^2yz.

19. a^2bc, ab^2c, abc^2. **20.** $2x^2y$, $3y^2z$, $4z^2x$. **21.** $p^5q^4r^3$, $p^4q^6r^7$, p^6q^2r.

22. $21a^2xy$, $24abx^2$, $28b^2yz^2$. **23.** $12l^3m^4$, $15m^4n^5$, $16n^5l^6$.

24. $5a(a-b)$ and $a^2(a+b)$. **25.** $3a+3b$ and $2(a+b)^2$.

26. $a+b$ and $(ca+cb)^2$.

Find the H.C.F. and L.C.M. of

27. $(a^2b)^3$, $(a^3b)^2$, $(ab^2)^4$. **28.** $(6a^2bc)^2$, $(4ab^2c)^3$, $(3abc^2)^4$.

29. $(a^2b^3)^4$, $(a^3b^4)^2$, $(a^4b^2)^3$. **30.** $(xy^2z^4)^2$, $(yz^3)^3$, $(z^2)^4$.

CHAPTER V.

THE DISTRIBUTIVE LAW.

33. Complete Statement of the Distributive Law.

If a, b, c stand for any three numbers, then

$$\mathbf{(a+b)c=ac+bc,} \quad \text{......(1)}$$
$$\mathbf{c(a+b)=ca+cb;} \quad \text{......(2)}$$

and if a is greater than b,

$$\mathbf{(a-b)c=ac-bc,} \quad \text{......(3)}$$
$$\mathbf{c(a-b)=ca-cb.} \quad \text{......(4)}$$

Identities (1) and (2) have been proved in Art. 12.

Proof of (3) and (4). If $a > b$,

$$a=(a-b)+b \qquad \text{(\textit{Def. of Subtraction}),}$$
$$\therefore\ ac=\{(a-b)+b\}c$$
$$\therefore\ ac=(a-b)c+bc. \qquad \text{(\textit{By identity} (i))}$$

Subtracting bc from each side (which is possible, for $ac > bc$),

$$ac-bc=(a-b)c.$$

Also, because $xy=yx$, it follows that

$$ca-cb=c(a-b).$$

The identities (1) and (3) are instances of the Distributive Law in its first form, which deals with a distribution of the multiplicand; the identities (2) and (4) are instances of the law in its second form, which deals with a distribution of the multiplier.

In the same way it can be shown that if x is any number and a, b, c are any numbers such that the operations indicated can be performed, then

$$(a+b-c)x=ax+bx-cx \text{ and } x(a+b-c)=xa+xb-xc,$$
$$(a-b-c)x=ax-bx-cx \text{ and } x(a-b-c)=xa-xb-xc.$$

Ex. 1. *Simplify* $5(2x-3y)-2(4x+y)$.

$$\begin{aligned}5(2x-3y)-2(4x+y) &= (10x-15y)-(8x+2y) && (\textit{Distributive Law})\\ &= 10x-15y-8x-2y\,* && (\textit{Associative Law})\\ &= 10x-8x-15y-2y && (\textit{Commutative Law})\\ &= (10x-8x)-(15y+2y) && (\textit{Associative Law})\\ &= 2x-17y.*\end{aligned}$$

NOTE. In practice, when the meaning of the operations is understood, all the steps except those marked * may be omitted.

Ex. 2. *Factorize the following expressions:*

(i) $3x^3yz^3+6x^2y^2z^3-9x^2yz^4$. (ii) $(a+b)^2-a-b$.

(i) $3x^3yz^3+6x^2y^2z^3-9x^2yz^4=3x^2yz^3(x+2y-3z)$.

(ii) $(a+b)^2-a-b=(a+b)^2-(a+b)=(a+b)(a+b-1)$.

EXERCISE X.

NOTE. *Letters, the values of which are not given, are supposed to have such values that the operations indicated are possible.*

State the values of

1. $112\times7-12\times7$. **2.** $39^2-39\times29$. **3.** $28\times27-27^2$.

4. $3\times98+3\times2$. **5.** $13\times16+5\times16+32$.

Factorize the following expressions:

6. $7+14x$. **7.** $10x-5y$. **8.** $2a^2-ab$. **9.** a^3-a^2.

10. $2a^4-2a^3$. **11.** $3ab+3a^2b^2$. **12.** a^2b+ab^2. **13.** $5a^3b^2-5a^2b^3$.

14. $7ab^2-14ab$. **15.** $9x^9-3x^3$. **16.** $(3x)^2-3x$. **17.** $x^4y^3z^2+x^2y^3z^4$.

18. $x(yz)^2-(xy)^2z$. **19.** $a^2+ab+ac$. **20.** $8x^3+2x^2-4x$.

21. $x^3y^2z+xy^3z^2-x^2yz^3$. **22.** $5x^4y-10x^3y^2+10x^2y^3-5xy^4$.

23. $(a+b)c+(a+b)d$. **24.** $(a-b)c-(a-b)d$.

25. $x(x+a)-b(x+a)$. **26.** $x(a+b)+y(a+b)-z(a+b)$.

27. $x(x^2+1)-2(x^2+1)$. **28.** $x^2(x+1)+x+1$. **29.** $a(b+1)-b-1$.

30. $x(a-b)-a+b$. **31.** $(a-b)^2-c(a-b)$. **32.** $(a-b)^2-a+b$.

Remove the brackets in the following and simplify:

33. $3(x-y)-2(x-2y)$. **34.** $5(y+2x)-2(x+6y)$.

35. $x(2y+3z)-y(2x+z)+yz$.

36. $x(4y-z)+y(4z-x)+z(4x-y)-2(yz+zx+xy)$.

37. $4(xy^2+yz^2+zx^2)-x(4y^2-z^2)-y(4z^2-x^2)-z(4x^2-y^2)$

38. Add $2(a-3b-2c)$, $3(b-4c+2a)$, and $4(2c-a+5b)$.

39. Subtract $3(a-b)-4(b-c)$ from $5(a-c)-2(a-b+c)$;

40. Add $2a(a-2b-c)$, $b(4a-3b+5c)$, $c(2a-5b+c)$, and subtract a^2-4b^2 from the sum.

41. Add $(3a-2b)x+(a-3b)$ and $(3a+2b)x-(a+3b)$.

42. In the identity $(a-b)c=ac-bc$:

(i) Substitute $(x+y)$ for c and deduce the expanded form of

$$(a-b)(x+y).$$

(ii) Substitute $(x-y)$ for c and find the expanded form of

$$(a-b)(x-y).$$

(iii) Substitute $(a-b)$ for c and prove the identity

$$(a-b)^2=a^2-2ab+b^2.$$

(iv) Substitute $(a+b)$ for c and prove the identity

$$(a+b)(a-b)=a^2-b^2.$$

(v) Substitute $(a^2-2ab+b^2)$ for c and prove the identity

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

34. Important Identities. **If a, b, x, y stand for any numbers, then**

$$\mathbf{(a+b)(x+y)=ax+ay+bx+by;} \quad \ldots\ldots\ldots\ldots\mathbf{(1)}$$

and, if a is greater than b,

$$\mathbf{(a-b)(x+y)=ax+ay-bx-by.} \quad \ldots\ldots\ldots\ldots\mathbf{(2)}$$

Proof. Let $x+y=c$, then

$$\begin{aligned}(a+b)(x+y)&=(a+b)c\\&=ac+bc &&\textit{(Distributive Law)}\\&=a(x+y)+b(x+y)\\&=(ax+ay)+(bx+by) &&\textit{(Distributive Law)}\\&=ax+ay+bx+by. &&\textit{(Associative Law)}\end{aligned}$$

Also

$$\begin{aligned}(a-b)(x+y)&=(a-b)c\\&=ac-bc &&\textit{(Distributive Law)}\\&=a(x+y)-b(x+y)\\&=(ax+ay)-(bx+by) &&\textit{(Distributive Law)}\\&=ax+ay-bx-by. &&\textit{(Associative Law)}\end{aligned}$$

Again, *if* **x** *is greater than* **y**, it follows in a similar manner that

$$\mathbf{(a+b)(x-y)=ax-ay+bx-by;} \qquad \mathbf{(3)}$$

and, *if* **a** *is greater than* **b** *and* **x** *greater than* **y**, then

$$\mathbf{(a-b)(x-y)=ax-ay-bx+by.} \qquad \mathbf{(4)}$$

Ex. 1. *Multiply* $(x+3)$ *by* $(x+2)$.

First method, by distributing the multiplicand:

$$\begin{aligned}(x+3)(x+2) &= x(x+2)+3(x+2)\\ &= x^2+2x+3x+6\\ &= x^2+5x+6.\end{aligned}$$

Second method, by distributing the multiplier:

$$\begin{aligned}(x+3)(x+2) &= (x+3)x+(x+3)2\\ &= x^2+3x+2x+6\\ &= x^2+5x+6.\end{aligned}$$

Ex. 2. *Multiply* $(2x+3)$ *by* $(5x-6)$.

Distributing the multiplier, we have

$$\begin{aligned}(2x+3)(5x-6) &= (2x+3)(5x)-(2x+3)6\\ &= 10x^2+15x-12x-18\\ &= 10x^2+3x-18.\end{aligned}$$

Ex. 3. *Multiply* $a+b+c$ *by* $x+y+z$.

$$\begin{aligned}(a+b+c)(x+y+z) &= (a+b+c)x+(a+b+c)y+(a+b+c)z\\ &= ax+bx+cx+ay+by+cy+az+bz+cz.\end{aligned}$$

Ex. 4. *Multiply* $a+b-c$ *by* $x-y-z$.

$$\begin{aligned}(a+b-c)(x-y-z) &= (a+b-c)x-(a+b-c)y-(a+b-c)z\\ &= (ax+bx-cx)-(ay+by-cy)-(az+bz-cz)\\ &= ax+bx-cx-ay-by+cy-az-bz+cz.\end{aligned}$$

Ex. 5. *Multiply* $a^2-2ab+b^2$ *by* $a-b$.

$$\begin{aligned}(a^2-2ab+b^2)(a-b) &= (a^2-2ab+b^2)a-(a^2-2ab+b^2)b\\ &= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3\\ &= a^3-3a^2b+3ab^2-b^3.\end{aligned}$$

Ex. 6. *Expand the product* $(x^2-3x+2)(x^2-2x-3)$.

$$\begin{aligned}&(x^2-3x+2)(x^2-2x-3)\\ &\quad= (x^2-3x+2)x^2-(x^2-3x+2)(2x)-(x^2-3x+2)3\\ &\quad= x^4-3x^3+2x^2-2x^3+6x^2-4x-3x^2+9x-6\\ &\quad= x^4-5x^3+5x^2+5x-6.\end{aligned}$$

Ex. 7. *Prove that*

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (bc+ca+ab)x + abc,$$

$$\begin{aligned}(x+a)(x+b)(x+c) &= (x^2+ax+bx+ab)(x+c)\\ &= x^3+ax^2+bx^2+abx+cx^2+cax+bcx+abc\\ &= x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.\end{aligned}$$

35. Ascending and Descending Order. The terms of the expression $a^3+3a^2b+3ab^2+b^3$ are said to be arranged in **descending powers** of a, for the term containing the highest power of a (viz. a^3) is on the left; this term is followed by the term containing the next highest power of a, and so on in succession, the term which is *independent of a* (that is, which does not contain a) being on the extreme right.

In the same way $a^3+3a^2b+3ab^2+b^3$ is said to be arranged in **ascending powers** of b.

In performing multiplications, it will be found convenient to arrange the terms both in the multiplicand and in the multiplier in ascending powers of some common letter *or* in descending powers of a common letter.

EXERCISE XI.

Expand the following products

1. $(x+2)(x+5)$.

2. $(x-2)(x-5)$.

3. $(x+2)(x-5)$.

4. $(x-2)(x+5)$.

5. $(3x+1)(5x+1)$.

6. $(3x-1)(5x-1)$.

7. $(3x+1)(5x-1)$.

8. $(3x-1)(5x+1)$.

9. $(a+b)(a+b)$.

10. $(a-b)(a-b)$.

11. $(4x+y)(5x+y)$.

12. $(4x-y)(5x-y)$.

13. $(4x+y)(5x-y)$.

14. $(4x-y)(5x+y)$.

15. $(2x+3y)(4x+7y)$.

16. $(2x-3y)(4x-7y)$.

17. $(2x+3y)(4x-7y)$.

18. $(2x-3y)(4x+7y)$.

19. $(x^2+8)(x^2-5)$.

20. $(2xy-5)(3xy+6)$.

21. $(7x^2-8y^2)(x^2+y^2)$.

22. $(x^2+x+1)(x-1)$.

23. $(x^2-x+1)(x+1)$.

24. $(4x^2+2x+1)(2x-1)$.

25. $(9x^2-3x+1)(3x+1)$.

26. $(a^2+2ab+b^2)(a+b)$.

27. $(a^2+ab+b^2)(a-b)$.

28. $(a^2-ab+b^2)(a+b)$.

29. $(3x^2-5)(2x-4)$.
30. $(x^2+3x+2)(x-3)$.
31. $(x^3+2x^2+4x+8)(x-2)$.
32. $(x^2-2x-3)(x^2+5x+6)$.
33. $(2x^2+5x-12)(3x^2-16x+5)$.
34. $(a^2+2ab+b^2)(a^2+2ab+b^2)$.
35. $(a^2-2ab+b^2)(a^2-2ab+b^2)$.
36. $(a^2+2ab+b^2)(a^2-2ab+b^2)$.

36. Important identities. The following particular cases of identities proved in Art. 34 are of great importance:

$$(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\mathbf{ab}+\mathbf{b}^2, \qquad\text{(1)}$$

$$(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2-2\mathbf{ab}+\mathbf{b}^2, \qquad\text{(2)}$$

$$(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2. \qquad\text{(3)}$$

The above three results may be stated as follows:

(1) **The square of the sum of two numbers is equal to the sum of the squares of the numbers together with twice the product of the numbers.**

(2) **The square of the difference of two numbers is equal to the sum of the squares of the numbers diminished by twice the product of the numbers.**

(3) **The product of the sum and difference of two numbers is equal to the difference of the squares of the numbers.**

Ex. 1. *Find the expanded form of* $(x+y+z)^2$.

Substituting x for a and $(y+z)$ for b in the identity

$$(a+b)^2=a^2+2ab+b^2,$$

we have

$$\begin{aligned}(x+y+z)^2&=x^2+2x(y+z)+(y+z)^2\\&=x^2+2xy+2xz+y^2+2yz+z^2\\&=x^2+y^2+z^2+2yz+2zx+2xy.\end{aligned}$$

Ex. 2. *Expand* $(x-1)^4$.

Since $(x-1)^4$ is the square of $(x-1)^2$

and $\quad (x-1)^2=x^2-2x+1=x^2-(2x-1)$,

we have

$$\begin{aligned}(x-1)^4&=\{x^2-(2x-1)\}^2\\&=x^4-2x^2(2x-1)+(2x-1)^2\\&=x^4-4x^3+2x^2+4x^2-4x+1\\&=x^4-4x^3+6x^2-4x+1.\end{aligned}$$

Other important identities are

$$\mathbf{(a+b)^3=a^3-3a^2b+3ab^2+b^3}, \quad \ldots\ldots\ldots\ldots(4)$$

$$\mathbf{(a-b)^3=a^3-3a^2b+3ab^2-b^3}, \quad \ldots\ldots\ldots\ldots(5)$$

$$\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2)}, \quad \ldots\ldots\ldots\ldots(6)$$

$$\mathbf{a^3-b^3=(a-b)(a^2+ab+b^2)}. \quad \ldots\ldots\ldots\ldots(7)$$

Ex. 3. *Expand* $(x^3+2x^2-3x-4)(x^3-2x^2+3x-4)$.

$$\begin{aligned}\text{The product} &= \{(x^3-4)+(2x^2-3x)\}\{(x^3-4)-(2x^2-3x)\} \\ &= (x^3-4)^2-(2x^2-3x)^2 && (\textit{By identity } 3) \\ &= (x^6-8x^3+16)-(4x^4-12x^3+9x^2) && (\textit{By identity } 2) \\ &= x^6-4x^4+4x^3-9x^2+16.\end{aligned}$$

Ex. 4. *Simplify* $(3x+2y)^2-2(3x+2y)(2x-3y)+(2x-3y)^2$.

Substituting a for $(3x+2y)$ and b for $(2x-3y)$,

$$\begin{aligned}\text{the expression} &= a^2-2ab+b^2 \\ &= (a-b)^2 \\ &= \{(3x+2y)-(2x-3y)\}^2 \\ &= (x+5y)^2 \\ &= x^2+10xy+25y^2.\end{aligned}$$

37. An Important Theorem. The square of the sum of several numbers is equal to the sum of the squares of the numbers together with the sum of twice the products of the numbers, taken two together in all possible ways.

Let the numbers be denoted by $a, b, c, d \ldots$, then

$$\begin{aligned}(a+b+c+d+\ldots)^2 &= a^2+b^2+c^2+d^2+\ldots+2ab+2ac \\ &\quad +2ad+2bc+2bd+2cd+\ldots.\end{aligned}$$

Proof.
$$\begin{aligned}(a+b+c+\ldots)^2 &= (a+b+c+\ldots)(a+b+c+\ldots) \\ &= (a+b+c+\ldots)a+(a+b+c+\ldots)b \\ &\quad +(a+b+c+\ldots)c+\ldots \quad \ldots\ldots\ldots(\alpha).\end{aligned}$$

Consider the product formed by taking any two of the letters, say bc: this product occurs twice in the expression (α), namely in $(a+b+c+\ldots)b$ and in $(a+b+c+\ldots)c$. Thus the expression (α) consists of the sum of the squares of $a, b, c \ldots$ and twice the products of $a, b, c \ldots$, two together. Hence the result follows.

NOTE. *The easiest way of writing down all the products two together of a, b, c, d is to take a with every letter which follows a, b with every letter which follows b and so on.*

EXERCISE XII.

1. In each of the formulae

$$(a+b)^2=a^2+2ab+b^2 \text{ and } (a-b)^2=a^2-2ab+b^2,$$

substitute

(i) $2x$ for a, 1 for b. (ii) $5x$ for a, $3y$ for b.
(iii) x^2 for a, y^2 for b. (iv) $3x^3$ for a, y^3 for b.
(v) x for a, $(y-z)$ for b. (vi) $2x$ for a, $(3y-5z)$ for b.

2. Write down the squares of the following:

(i) $x-1$. (ii) $x-4$. (iii) $x+5$.
(iv) $3x-1$. (v) $3x-2$. (vi) $5x+4y$.
(vii) $6x-7y$. (viii) $7x+8y$. (ix) $7xy-2$.
(x) $3(5x+7)$. (xi) x^2-y^2. (xii) $6x^3-1$.

3. Of what expressions are the following the squares? Verify the results by substituting 1 for x and 1 for y:

(i) $4x^2+20x+25$. (ii) $9x^2-12x+4$.
(iii) $16x^2-8xy+y^2$. (iv) $4x^4-4x^2y+y^2$.
(v) $36x^6+60x^3y^3+25y^6$. (vi) $(2x+5y)^2+1-4x-10y$.

4. By making the proper substitutions in the identity

$$(a+b)(a-b)=a^2-b^2,$$

expand the following products

(i) $(x+1)(x-1)$. (ii) $(x+5)(x-5)$. (iii) $(2x+3)(2x-3)$.
(iv) $(4x-5)(4x+5)$. (v) $(5x+6y)(5x-6y)$.
(vi) $(x^2+y^2)(x^2-y^2)$. (vii) $(3x^3+y^3)(3x^3-y^3)$.
(viii) $(ab+2)(ab-2)$. (ix) $(a^4+b^4)(a^4-b^4)$.

5. In the identity $(a-b)^2=a^2-2ab+b^2$:

(i) By substituting x for a and $(y+z)$ for b, find the expanded form of $(x-y-z)^2$.

(ii) By substituting x for a and $(y-z)$ for b, find the expanded form of $(x-y+z)^2$.

6. Write down the squares of

(i) $2x+3y-z$. (ii) $2x-3y-4z$. (iii) $x+2y+5z$.
(iv) $x^2-4y^2+z^2$. (v) x^2+x+1. (vi) $2x^2-3x-1$.
(vii) $3x^2-4x+5$. (viii) x^3+x^2-x-1.

7. In each of the formulae

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3,$$

make the substitutions (i)–(iv) of Ex. 1.

8. Write down the cubes of the expressions (i) (vii) in Ex. 2.

9. In each of the identities

$a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$,

make the substitutions (i)–(iv) of Ex. 1.

10. By making the proper substitutions in certain formulae, find the values of

(i) $7^2 + 2.7.3 + 3^2$. (ii) $7^2 - 2.7.6 + 6^2$.

(iii) $8^3 + 3.8^2.2 + 3.8.2^2 + 2^3$. (iv) $9^3 + 3.9^2 + 3.9 + 1$.

(v) $5^3 - 3.5^2.4 + 3.5.4^2 - 4^3$.

Expand the following products

11. $(x^2+x+1)(x^2-x+1)$.

12. $[ax-(a-2)][ax-(a+2)]$.

13. $(x^2+3xy+y^2)(x^2-3xy+y^2)$.

14. $(x^2-xy+2y^2)(x^2+xy-2y^2)$.

15. $(a^2-2ab+2b^2)(a^2+2ab+2b^2)$.

16. $(x^3+2x^2+3x+4)(x^3-2x^2-3x+4)$.

17. $(x^2-ax+bx-ab)(x^2+ax-bx-ab)$.

18. Multiply x^2-3x+3 by x^2+3x+3.

19. Multiply $x^3-3x^2y-3xy^2$ by $x^2-5xy+2y^2$.

Verify by substituting 5 for x and 1 for y.

20. Find the continued product of

$$x^2+xy+y^2,\ x^2-xy+y^2, \text{ and } x^4-x^2y^2+y^4.$$

Simplify the following

21. $(x+1)^2-2x(x+1)+x^2$. 22. $(x+y)^2-2(x+y)(x-y)+(x-y)^2$.

23. $(x+y)^2+2(x+y)(x-y)+(x-y)^2$.

24. $(x+2y)^2-(x+2y)(x-2y)+(x-2y)^2$.

25. $(2x+y)^3-6x(2x+y)^2+12x^2(2x+y)-8x^3$.

26. $(2x+y)^3+3(2x+y)^2(x+2y)+3(2x+y)(x+2y)^2+(x+2y)^3$.

27. $(x+1)(x-1)(x^2+x+1)(x^2-x+1)$.

28. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.

29. $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$.

30. $(x+1)(x-2)-2(x+2)(x-3)+(x+3)(x-1)$.

31. $2x(x+3y)-3y(2x-y)-(x-y)(2x-3y)$.

32. $3x(2x-5y)-2y(5x-3y)-(6x-y)(x-6y)$.

33. $(a-b)(a+2b)-(a-2x)(a+x)-(a-2x+2b)(a-x-b)$.

34. $(x+2y+z)(x+2y-z)-(x-2y+z)(x-2y-z)$.

35. $(a-b)(x+a)(x-b)-a(x-b)^2+b(x+a)^2$.

36. $(x^2+xy+y^2)^2-(x^2-y^2)^2-xy(2x-y)(x-2y)$.

Prove the following identities :

37. $(a-b+c-d)^2+(a-b-c+d)^2=2(a-b)^2+2(c-d)^2$.

38. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$
$$=\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}$$
$$=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

39. If $a^2+b^2=c^2$ simplify the expression
$$(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

40. A room is a feet long, b feet broad, and there is a stained border, 3 feet wide, round the floor. Find, in square feet, the area of the unstained part of the floor.

41. A square garden a yards long has a path b feet wide all round. Find, in square feet, the area of the path.

42. A box is made of wood 1 in. thick and the box has a lid of the same thickness: find the number of cubic inches of wood required to make the box when the external dimensions (including lid) are as follows: (i) length, breadth and height each a inches; (ii) length, breadth and height respectively a, b, c inches.

43. Find in square inches the area (i) of the outside, (ii) of the inside of the box and lid in each of the cases in Ex. 42.

CHAPTER VI.

FACTORS.

NOTE. *In Exercises XIII.–XX. the student is advised to work only the earlier examples on first reading.*

38. Factors of Algebraical Expressions. It is always possible to find the product of two given algebraical expressions, but the converse problem, to resolve a given algebraical expression into factors, is not always possible.

In searching for the factors of an algebraical expression, we are guided (in the first instance) by our experience in multiplication.

The case of an expression whose terms have a common factor has been considered in Arts. 12 and 33; other methods of factorizing algebraical expressions will now be given

39. Method of Grouping Terms.

Ex. *Resolve into factors*

(i) $ax + ay - bx - by$. (ii) $x^2 - x(a - b) - ab$.

(iii) $ax - ay + a - bx + by - b$. (iv) $b^2 - 2bc + c^2 - 3b + 3c$.

(v) $a^2 - b^2 - 3ax - 3bx$.

(i)
$$\begin{aligned} ax + ay - bx - by &= (ax + ay) - (bx + by) \\ &= a(x + y) - b(x + y) \\ &= (a - b)(x + y). \end{aligned}$$

(ii) $$\begin{aligned} x^2 - x(a-b) - ab &= x^2 - xa + xb - ab \\ &= (x^2 - xa) + (xb - ab) \\ &= x(x-a) + b(x-a) \\ &= (x+b)(x-a). \end{aligned}$$

(iii) $$\begin{aligned} ax - ay + a - bx + by - b &= (ax - ay + a) - (bx - by + b) \\ &= a(x - y + 1) - b(x - y + 1) \\ &= (a-b)(x-y+1). \end{aligned}$$

(iv) $$\begin{aligned} b^2 - 2bc + c^2 - 3b + 3c &= (b^2 - 2bc + c^2) - (3b - 3c) \\ &= (b-c)^2 - 3(b-c) \\ &= (b - c - 3)(b-c). \end{aligned}$$

(v) $$\begin{aligned} a^2 - b^2 - 3ax - 3bx &= (a^2 - b^2) - (3ax + 3bx) \\ &= (a-b)(a+b) - 3x(a+b) \\ &= (a - b - 3x)(a+b). \end{aligned}$$

EXERCISE XIII.

Express the following as the product of factors by grouping terms:

1. $ab + bc + ca + c^2$.
2. $ab - bc + ca - c^2$.
3. $ab - bc - ca + c^2$.
4. $a^2 + 2a + ab + 2b$.
5. $x^2 + xy - 3x - 3y$.
6. $3x - 3y - 2zx + 2yz$.
7. $xy + 2x - y - 2$.
8. $xy - 5x - 3y + 15$.
9. $2xy - 3ax + 10ay - 15a^2$.
10. $x^2 + (a+b)x + ab$.
11. $x^2 + (a-b)x - ab$.
12. $x^2 - (a+b)x + ab$.
13. $y^2 - xy + 9y - 9x$.
14. $a^3 - a^2 - a + 1$
15. $x^3 + x^2y + xy^2 + y^3$.
16. $x^2 - y^2 + xz - yz$.
17. $a(x^2 + y^2) + (a^2 + 1)xy$.
18. $ab(l^2 + m^2) + lm(a^2 + b^2)$.
19. $a^2 + b^2 + 2ab + 2bc + 2ca$.
20. $a^2 - b^2 + 2bc + 2ca$.
21. $a^2 - b^2 + 2bc - 2ca$.
22. $y^2 - abx^4 + (b-a)x^2y$.
23. $ax^2 + by^2 + byz + azx + (a+b)xy$.
24. $5y - abx - 5ax + by$.
25. $5b - abx - 5ax + a^2x^2$.
26. $a^2(b+c) + b^2(c+a) + c^2(a+b) + 3abc$.

[In Ex. 26, group thus: $(a^2b + ab^2 + abc) + \ldots$.]

27. $6(x^2+2)yz + x(8y^2 + 9z^2)$.
28. $a(a+c) - b(b+c)$.
29. $(x+a)(x+b)(x+c) - (x+a)bc$.
30. $(a+2b)^3 - a^2c - 4b^2c - 4abc$.
31. $(3a-2b)^3 - (18a^2 + 8b^2) + 24ab$.

40. Factors of $x^2 \pm px + q$.* If p and q stand for given numbers, and if two numbers a and b can be found whose product is q and whose sum is p, then

$$\begin{aligned} x^2+px+q &= x^2+x(a+b)+ab \\ &= x^2+xa+xb+ab \\ &= (x+a)x+(x+a)b \\ &= (x+a)(x+b). \quad \ldots\ldots\ldots\ldots(\alpha) \end{aligned}$$

In a similar way it can be shown that if a and b have the same values as before

$$x^2-px+q=(x-a)(x-b). \quad \ldots\ldots\ldots\ldots(\beta)$$

* $\pm$ is an abbreviation for "plus or minus."

Ex. 1. *Factorize* $x^2+7x+12$.

Here the signs are all $+$, and we write

$$x^2+7x+12=(x+?)(x+?),$$

leaving blanks in the places marked in the text by notes of interrogation. These blanks are to be filled with two numbers (if such can be found), whose product is 12 and whose sum is 7 ; these are 3, 4, and as in (α)

$$x^2+7x+12=(x+3)(x+4).$$

Ex. 2. *Factorize* $x^2-19x+18$.

Here the sign of the third term is $+$ and that of the middle term is $-$, and we write $x^2-19x+18=(x-?)(x-?)$, where the blanks are to be filled with two numbers whose product is 18 and whose sum is 19 ; these are 18 and 1, and as in (β)

$$x^2-19x+18=(x-18)(x-1).$$

Ex. 3. *Factorize* $x^2-16xy+63y^2$.

We write $x^2-16xy+63y^2=(x-?y)(x-?y)$, where the blanks are to be filled with two numbers whose product is 63 and whose sum is 16 ; these are 9, 7, so that

$$x^2-16xy+63y^2=(x-9y)(x-7y).$$

Ex. 4. *Factorize* $(x+y)^2+3(x+y)z+2z^2$.

As in Ex. 3, we have

$$\begin{aligned} (x+y)^2+3(x+y)z+2z^2 &= \{(x+y)+z\}\{(x+y)+2z\} \\ &= (x+y+z)(x+y+2z). \end{aligned}$$

EXERCISE XIV.

Expand the following products :

1. $(x+1)(x+2)$.
2. $(x-3)(x-5)$.
3. $(x-5)(x-7)$.
4. $(x+5)(x+12)$.
5. $(x-4)(x-7)$.
6. $(5+x)(5+x)$.
7. $(x-2y)(x-3y)$.
8. $(x+5y)(x+6y)$.
9. $(6x+y)(7x+y)$.
10. $(8x-y)(10x-y)$.
11. $(x^2-4)(x^2-6)$.
12. $(x^3-7)(x^3-8)$.

Factorize the following expressions :

13. x^2+4x+3.
14. x^2-5x+6.
15. x^2-5x+4.
16. x^2+4x+4.
17. x^2-6x+8.
18. x^2-9x+8.
19. x^2+6x+9.
20. $x^2+10x+9$.
21. $x^2+10x+16$.
22. $x^2+10x+21$.
23. $x^2+10x+24$.
24. $x^2+10x+25$.
25. $x^2-11xy+30y^2$.
26. $x^2-11xy+28y^2$.
27. $x^2-11xy+24y$.
28. $x^2-11xy+18y^2$.
29. $x^2-11xy+10y^2$.
30. $20x^2+21xy+y^2$.
31. $20x^2+12xy+y^2$.
32. $20x^2+9xy+y^2$.
33. $28+29x^2+x^4$.
34. $28+16x^2+x^4$.
35. $28+11x^2+x^4$.
36. $x^2y^2-25xy+24$.
37. $x^2y^2-14xy+24$.
38. $x^2y^2-11xy+24$.
39. $x^2y^2-10xy+24$.
40. x^6+41x^3+40.
41. x^6+22x^3+40.
42. x^6+14x^3+40.
43. x^6+13x^3+40.
44. $x^4+31x^2y+30y^2$.
45. $x^4+17x^2y+30y^2$.
46. $x^4+13x^2y+30y^2$.
47. $x^4+11x^2y+30y^2$.
48. x^8+20x^4+100.
49. $(x+y)^2-6z(x+y)+5z^2$.
50. $z^2-6z(x+y)+5(x+y)^2$.
51. $(x-y)^2-7z(x-y)+10z^2$.
52. $z^2-7z(x-y)+10(x-y)^2$.

41. Factors of $x^2 \pm px - q$. If p and q stand for given numbers, and if two numbers a and b can be found such that $q=ab$ and $p=a-b$ (a being greater than b), then

$$\begin{aligned} x^2+px-q &= x^2+(a-b)x-ab \\ &= x^2+ax-bx-ab \\ &= (x^2+ax)-(bx+ab) \\ &= (x+a)x-(x+a)b \\ &= (x+a)(x-b), \quad \ldots\ldots\ldots\ldots\ldots(\alpha) \end{aligned}$$

and in a similar way

$$x^2-px-q=(x-a)(x+b). \quad \ldots\ldots\ldots\ldots\ldots(\beta)$$

With regard to the arrangement of signs, observe that in both (α) and (β)

(i) The sign which precedes q is $-$, and the signs which precede a and b are unlike (that is, one sign is $+$ and the other is $-$).

(ii) The sign which precedes the greater of the numbers a and b is the same as that which precedes px.

Ex. 1. *Factorize the expression* x^2+x-12.

Because the sign of the third term is $-$ we write

$$x^2+x-12=(x+?)(x-?),$$

where the blanks are to be filled with two numbers whose product is 12 and whose difference is 1; these are 4, 3. Since the sign of the middle term of the given expression is $+$, 4 must follow $+$, thus:

$$x^2+x-12=(x+4)(x-3).$$

Ex. 2. *Resolve into factors* $x^2-3x-504$.

We write $x^2-3x-504=(x-?)(x+?)$, where the blanks are to be filled with two numbers (if such exist) whose product is 504 and whose difference is 3.

The process of search will be facilitated by expressing 504 as the product of prime factors, thus:

$$504=2^3 \,.\, 3^2 \,.\, 7.$$

We may write down all the pairs of numbers whose product is 504, thus:

$$(1\times 504),\ (2\times 252),\ (3\times 168)\ \ldots .$$

This is unnecessary, for the required numbers are nearly equal, differing only by 3. Noting that as the pairs are written down in succession, the difference diminishes, we try a fairly large factor of 504, say 18; now $18\times 28=504$ and $28-18=10$, so that the difference is still too large. Try the numbers 19, 20, 21 ... to find larger factors.

We find $21\times 24=504$ and $24-21=3$, so that 21 and 24 are the required numbers.

Since the sign of the middle term is $-$, 24 must be preceded by $-$, and

$$x^2-3x-504=(x-24)(x+21).$$

A shorter method.—Observing that 3^2 is a factor of 504 we write

$$x^2-3x-504=x^2-x\times 3-56\times 3^2.$$

We now seek two numbers whose product is 56 and whose difference is 1; these are 7 and 8, so that

$$x^2-x.3-56.3^2=(x-8.3)(x+7.3);$$

$$\therefore\ x^2-3x-504=(x-24)(x+21).$$

The artifice employed in this example is of frequent use.

Ex. 3. *Factorize* $10(x+y)^2-3(x+y)(x-y)-(x-y)^2$.

Substituting a for $(x+y)$ and b for $(x-y)$,

$$\text{the expression}=10a^2-3ab-b^2$$
$$=(5a+b)(2a-b)$$

Now
$$5a+b=5(x+y)+(x-y)=6x+4y,$$
$$2a-b=2(x+y)-(x-y)=x+3y;$$

$$\therefore\ \text{the expression}=(6x+4y)(x+3y)=2(3x+2y)(x+3y).$$

EXERCISE XV.

Expand the following products:

1. $(x-1)(x+2)$. **2.** $(x+1)(x-2)$. **3.** $(x-3)(x+7)$.

4. $(x+3)(x-7)$. **5.** $(x+6y)(x-8y)$. **6.** $(x-5y)(x+10y)$.

7. $(x^2-8)(x^2+3)$. **8.** $(x^3-4)(x^3+9)$. **9.** $(xy-8)(xy+12)$.

Factorize the following expressions:

10. x^2+2x-3. **11.** x^2+2x-8. **12.** $x^2+2x-15$.

13. x^2-3x-4. **14.** $x^2-3x-10$. **15.** $x^2-3x-18$.

16. $x^2+11x-12$. **17.** $x^2-11x-12$. **18.** $x^2-4x-12$.

19. $x^2+4x-12$. **20.** x^2+x-12. **21.** x^2-x-12.

22. $x^2+23x-24$. **23.** $x^2-10x-24$. **24.** $x^2-5x-24$.

25. $x^2+2x-24$. **26.** $x^2+17x-18$. **27.** $x^2-7x-18$.

28. $x^2-3x-18$. **29.** $x^2+39xy-40y^2$. **30.** $x^2-18xy-40y^2$.

31. $x^2-6xy-40y^2$. **32.** $x^2+3xy-40y^2$. **33.** x^6-27x^3-28.

34. x^6-12x^3-28. **35.** x^6+3x^3-28. **36.** x^4-29x^2-30.

37. x^4-13x^2-30. **38.** x^4+7x^2-30. **39.** x^4+x^2-30.

40. $x^2-5x-150$. **41.** $x^2+8x-240$. **42.** $x^2-28x-245$.

43. $x^2-20xy^2-300y^4$. **44.** $x^2-9xy^3-360y^6$. **45.** $x^2+20x-224$.

46. $x^2+2x-264$. **47.** x^2 [illegible] $x-240$. **48.** $x^2-4xy^4-480y^8$.

49. $x^2+6xy-112y^2$. **50.** $x^2-6xy^6-315y^{12}$. **51.** $x^2-12xy-405y^2$.

52. $(x+y)^2+2(x+y)(x-y)-8(x-y)^2$.

53. $(3x+y)^2+2(3x+y)(x+3y)-3(x+3y)^2$.

54. $(5x+y)^2+4(5x+y)(5x-y)-5(5x-y)^2$.

55. $(x+y+z)^2+3z(x+y+z)-4z^2$. **56.** $(x+y+2z)^2+z(x+y+2z)-6z^2$.

42. Factors of $ax^2 \pm bx+c$. If p, q, l, m stand for any numbers

$$\begin{aligned}(px+q)(lx+m)&=(px+q)lx+(px+q)m \quad\ldots\ldots(\alpha)\\ &=plx^2+qlx+pmx+qm \quad\ldots\ldots(\beta)\\ &=plx^2+(ql+pm)x+qm \quad\ldots\ldots(\gamma)\\ &=ax^2+bx+c, \quad\ldots\ldots(\delta)\end{aligned}$$

where (for shortness) a stands for pl, b for $(ql+pm)$ and c for qm.

Also
$$\begin{aligned}(px-q)(lx-m)&=(px-q)lx-(px-q)m \quad\ldots\ldots(\alpha)\\ &=plx^2-qlx-pmx+qm \quad\ldots\ldots(\beta)\\ &=plx^2-(ql+pm)x+qm \quad\ldots\ldots(\gamma)\\ &=ax^2-bx+c, \quad\ldots\ldots(\delta)\end{aligned}$$

where a, b, c have the same values as before.

Now let a, b, c stand for given numbers and let us try to factorize the expressions $ax^2 \pm bx+c$.

First method. If we try to reverse the order of the steps in the above work, the difficulty lies in proceeding from (δ) to (γ). To do this, b must be replaced by the sum of two numbers ql and pm whose product is $pl \times qm$, that is ac, and whose sum is b; we are therefore led to the following rule:

To factorize $\mathbf{ax^2 \pm bx+c}$, *replace* **b** *by the* **sum** *of two numbers whose product is* **ac** *and whose sum is* **b**; *the given expressions can then be factorized by grouping the terms.*

Ex. 1. *Factorize* $6x^2+17x+12$.

We seek two numbers whose product is 6×12 (or 72) and whose sum is 17; these are 8 and 9, and

$$\begin{aligned}6x^2+17x+12&=6x^2+(9+8)x+12\\ &=6x^2+9x+8x+12\\ &=3x(2x+3)+4(2x+3)\\ &=(3x+4)(2x+3).\end{aligned}$$

Ex. 2. *Factorize* $60x^2 - 77x + 24$.

We seek two numbers whose product is 60×24 (that is, $2^5 . 3^2 . 5$), and whose sum is 77. Try the numbers 60 and 24; their sum is too great. Try the numbers 25, 26, 27, ... in succession to see which of these are factors of 60×24. Write these numbers down and cross out those which are not factors of 60×24, thus

$$\cancel{25},\ \cancel{26},\ \cancel{27},\ \cancel{28},\ \cancel{29},\ 30,\ \cancel{31},\ 32, \dots.$$

Now $\qquad 60 \times 24 = 30 \times 48 = 32 \times 45,$

and $\qquad 32 + 45 = 77;$

thus 32 and 45 are the required numbers and

$$\begin{aligned} 60x^2 - 77x + 24 &= 60x^2 - 45x - 32x + 24 \\ &= (60x^2 - 45x) - (32x - 24) \\ &= 15x(4x - 3) - 8(4x - 3) \\ &= (4x - 3)(15x - 8). \end{aligned}$$

Second method: as described in the following example.

Ex. 3. *Factorize* $4x^2 - 9x + 2$.

Observing that the sign of the last term is + and that of the middle term −, we write

$$4x^2 - 9x + 2 = (?x - ?)(?x - ?),$$

where the blanks to the left of x are to be filled with two numbers whose product is 4, and the other blanks with two numbers whose product is 2. The possible trial-arrangements are,

(i) $(x - 1)(4x - 2)$; (ii) $(4x - 1)(x - 2)$; (iii) $(2x - 1)(2x - 2)$;

of these, (i) and (iii) may be discarded at once, for both $4x - 2$ and $2x - 2$ contain the factor 2, which is not a factor of $4x^2 - 9x + 2$. To see if the remaining arrangement is correct, the multiplication must be performed, so far as is necessary to find the middle term. Doing this (mentally), it is seen that the middle term is $-9x$, thus

$$4x^2 - 9x + 2 = (4x - 1)(x - 2).$$

Ex. 4. *Factorize* $28x^2y^2 - 63xy^3 + 14y^4$.

$$\begin{aligned} 28x^2y^2 - 63xy^3 + 14y^4 &= 7y^2(4x^2 - 9xy + 2y^2) \\ &= 7y^2(4x - y)(x - 2y). \end{aligned}$$

Ex. 5. *Factorize* $2x^2+35x+75$.

Here 5^2 *is a factor of 75 and 5 is a factor of 35, and the easiest method is as follows:*

$$2x^2+35x+75 = 2x^2+7x\,.\,5+3\,.\,5^2$$
$$=(2x+1\,.\,5)(x+3\,.\,5)$$
$$=(2x+5)(x+15).$$

Ex. 6. *Factorize* $2x^2+31x+75$.

Here 5^2 *is a factor of 75 and 5 is not a factor of 31*, and it is easy to see that 5^2 must be a factor of the second term of one of the required factors. We therefore at once try the arrangement $(2x+25)(x+3)$, which proves to be correct.

EXERCISE XVI.

Expand the following products:

1. $(x+1)(2x+3)$.
2. $(x+3)(2x+1)$.
3. $(x+4)(3x+1)$.
4. $(x+2)(3x+2)$.
5. $(x+1)(3x+4)$.
6. $(x-6)(5x-1)$.
7. $(x-3)(5x-2)$.
8. $(x-2)(5x-3)$.
9. $(x-1)(5x-6)$.
10. $(2x+3)(3x+2)$.
11. $(2x^2+1)(3x^2+5)$.
12. $(2x^2+5)(3x^2+1)$.
13. $(6x^2+1)(x^2+5)$.
14. $(6x^2+5)(x^2+1)$.
15. $(2x-3y)(5x-7y)$.
16. $(5x-3y)(2x-7y)$.
17. $(10x-3y)(x-7y)$.
18. $(x-3y)(10x-7y)$.
19. $(2x-21y)(5x-y)$.
20. $(2x-y)(5x-21y)$.

Expand the following products and bracket the terms containing x:

21. $(x+1)(ax+b)$.
22. $(ax+1)(x+b)$.
23. $(2x-1)(ax-b)$.
24. $(2x-b)(ax-1)$.
25. $(2ax+b)(3cx+d)$.

Factorize the following:

26. $3x^2+5x+2$.
27. $3x^2-5x+2$.
28. $3x^2-7x+2$.
29. $3x^2+9x+6$.
30. $3x^2+19x+6$.
31. $3x^2+11x+6$.
32. $2x^2+7x+3$.
33. $2x^2+5x+3$.
34. $2x^2-7x+6$.
35. $2x^2-13x+6$.
36. $3x^2+8x+4$.
37. $3x^2+7x+4$.

38. $3x^2+13x+4$. **39.** $5x^2-8x+3$. **40.** $5x^2-16x+3$.
41. $3x^2-14x+15$. **42.** $3x^2-46x+15$. **43.** $3x^2-11x+10$.
44. $3x^2-17x+10$. **45.** $3x^2-31x+10$. **46.** $3x^2-13x+10$.
47. $5x^2+17x+6$. **48.** $5x^2+13x+6$. **49.** $5x^2+11x+6$.
50. $5x^2+31x+6$. **51.** $6x^2+17x+7$. **52.** $6x^2+43x+7$.
53. $21x^2-17xy+2y^2$. **54.** $21x^2-23xy+2y^2$. **55.** $15x^2+38xy+7y^2$.
56. $15x^2+106xy+7y^2$. **57.** $10x^2-19x+7$. **58.** $10x^2-71x+7$.
59. $4x^4-12x^2+5$. **60.** $4x^4-9x^2+5$. **61.** $10x^2y^2-27xy+18$.
62. $10x^2y^2-63xy+18$. **63.** $10x^2+29x+21$. **64.** $10x^2+37x+21$.
65. $30x^2+65x+10$. **66.** $63x^2-69x+18$. **67.** $60x^2+148x+80$.
68. $20x^2+49x+30$. **69.** $14x^2-65x+56$. **70.** $16x^2-14x+3$.
71. $98x^2+49x+5$. **72.** $72x^2-42x+5$. **73.** $242x^2-77x+6$.
74. $600x^2+410x+63$. **75.** $162x^2-81x+10$.

43. Factors of $ax^2 \pm bx - c$. If p, q, l, m stand for any numbers such that $ql > pm$,

$$\begin{aligned}(px+q)(lx-m) &= (px+q)\,lx-(px+q)\,m \qquad (\alpha)\\ &= plx^2+qlx-pmx-qm \qquad (\beta)\\ &= plx^2+(ql-pm)x-qm \qquad (\gamma)\\ &= ax^2+bx-c, \qquad (\delta)\end{aligned}$$

where (for shortness) a stands for pl, b for $(ql-pm)$ and c for qm.

Also
$$\begin{aligned}(px-q)(lx+m) &= (px-q)\,lx+(px-q)\,m \qquad (\alpha)\\ &= plx^2-qlx+pmx-qm. \qquad (\beta)\\ &= plx^2-(ql-pm)x-qm \qquad (\gamma)\\ &= ax^2-bx-c, \qquad (\delta)\end{aligned}$$

where a, b, c have the same values as before. Now let a, b, c stand for given numbers and let us try to factorize the expressions $ax^2 \pm bx - c$.

First method. Observing that ql and pm are two numbers whose product is $pl \times qm$, that is ac, and whose difference is b, the correctness of the following rule will be evident:

To factorize $\mathbf{ax^2 \pm bx - c}$, *find two numbers whose product is* **ac** *and whose* **difference** *is* **b**; *if* **b** *is replaced by the difference of these numbers, the given expressions can then be factorized by grouping the terms.*

Ex. 1. *Factorize* $15x^2-2x-24$.

We seek two numbers whose product is 15×24 (or $2^3.3^2.5$) and whose difference is 2. Try the numbers 15 and 24; their difference is too large. Try 16, 17, 18 ... in succession, to see which are factors of 15×24. We find that $18\times 20=15\times 24$ and $20-18=2$, so that 18 and 20 are the required numbers and

$$\begin{aligned}15x^2-2x-24&=15x^2-20x+18x-24\\&=(15x^2-20x)+(18x-24)\\&=5x(3x-4)+6(3x-4)\\&=(3x-4)(5x+6).\end{aligned}$$

Second method: as described in the next example.

Ex. 2. *Factorize* $3x^2-11x-20$.

Observing that the sign of the last term is $-$, we write

$$3x^2-11x-20=(?x+?)(?x-?),$$

where the blanks to the left of x are to be filled with two numbers whose product is 3 and the other blanks with two numbers whose product is 20. The possible trial-arrangements are

$$\begin{array}{ll}(x+1)(3x-20), & \text{middle term}=-17x,\\(x+2)(3x-10), & \quad\text{,,}\quad\text{,,}\quad=-\ \ 4x,\\(x+4)(3x-5), & \quad\text{,,}\quad\text{,,}\quad=+\ \ 7x,\\(x+5)(3x-4), & \quad\text{,,}\quad\text{,,}\quad=+11x,\\(x+10)(3x-2), & \quad\text{,,}\quad\text{,,}\quad=+28x,\\(x+20)(3x-1), & \quad\text{,,}\quad\text{,,}\quad=+59x,\end{array}$$

and six other arrangements got from the above by changing the sign of the second term of each factor. To see which (if any) of these is correct, the multiplications must be performed (mentally) so far as is necessary to find the middle term in each case.

We find that

$$(x+5)(3x-4)=3x^2+11x-20,$$

and $$3x^2-11x-20=(x-5)(3x+4).$$

NOTE. In practice, it is unnecessary to write down all possible arrangements: as each is written down, the corresponding middle term should be calculated until the factors are found. The process is often facilitated by considerations like those in Ex. 3, 5, 6 of Art. 42.

EXERCISE XVII.

Expand the following products:

1. $(x-1)(2x+3)$.
2. $(x+1)(2x-3)$.
3. $(x+3)(2x-1)$.
4. $(x-3)(2x+1)$.
5. $(x-4)(3x+1)$.
6. $(x+2)(3x-2)$.
7. $(x-1)(3x+4)$.
8. $(2x+3)(3x-2)$.
9. $(2x+1)(3x-5)$.
10. $(2x+5)(3x-2)$.
11. $(2x-5y)(x+y)$.
12. $(3x-y)(x+5y)$.
13. $(6x^2-1)(x^2+5)$.
14. $(7x^3-2)(x^3+3)$.
15. $(5x+3)(2x-7)$.
16. $(4xy+5)(2xy-1)$.
17. $(x^2-3y)(7x^2+2y)$.
18. $(6x+7)(2x-3)$.

Factorize the following:

19. $3x^2-x-2$.
20. $3x^2+x-2$.
21. $3x^2+5x-2$.
22. $2x^2-9x-5$.
23. $2x^2-3x-5$.
24. $3x^2-7xy-6y^2$.
25. $3x^2+17xy-6y^2$.
26. $2x^2-5x-3$.
27. $2x^2-x-3$.
28. $3x^2+2x-8$.
29. $3x^2-5x-8$.
30. $5x^2+7x-6$.
31. $5x^2-29x-6$.
32. $15x^2+7xy-2y^2$.
33. $15x^2-13xy-2y^2$.
34. $5x^2+33x-14$.
35. $7x^4+48x^2-7$.
36. $6x^6-17x^3-10$.
37. $5x^2+4x-12$.
38. $5x^2+7x-12$.
39. $6x^2+11x-7$.
40. $14x^2-27x-2$.
41. $5x^2-8x-21$.
42. $5x^2-104x-21$.
43. $3x^2+16xy-12y^2$.
44. $3x^2+35x-12$.
45. $20x^2-9x-20$.
46. $12x^2-143x-12$.
47. $15x^4+32x^2y-7y^2$.
48. $6x^2-9x-15$.
49. $6x^2-7x-20$.
50. $6x^2+x-35$.
51. $24x^2+10x-25$.
52. $12x^2-13x-14$.
53. $10x^2-x-21$.
54. $8x^2+18x-35$.
55. $4x^2+37x-30$.
56. $32x^2+4x-3$.
57. $14x^2-33x-56$.
58. $98x^2+7x-6$.
59. $162x^2+27x-20$.
60. $50x^2-25x-42$.
61. $48x^2-8xy-16y^2$.
62. $6x^2-7xy-245y^2$.

44. Difference of two Squares. By means of the identity

$$a^2-b^2=(a+b)(a-b)$$

a large number of expressions can be resolved into factors: for this purpose the identity may be stated as follows:—

The difference of the squares of two numbers is equal to the product of the sum and difference of the numbers.

Ex. 1. *Express as the product of factors*

(i) $25x^4 - 9y^2z^2$; (ii) $9a^2 - 4(b-c)^2$;

(iii) $3x^7 - 147x$; (iv) $a^4 - 16b^4$.

(i) $25x^4 - 9y^2z^2 = (5x^2)^2 - (3yz)^2 = (5x^2 + 3yz)(5x^2 - 3yz)$,

(ii)
$$\begin{aligned} 9a^2 - 4(b-c)^2 &= (3a)^2 - \{2(b-c)\}^2 \\ &= \{3a + 2(b-c)\}\{3a - 2(b-c)\} \\ &= (3a + 2b - 2c)(3a - 2b + 2c), \end{aligned}$$

(iii)
$$\begin{aligned} 3x^7 - 147x &= 3x(x^6 - 49) \\ &= 3x\{(x^3)^2 - 7^2\} \\ &= 3x(x^3 + 7)(x^3 - 7), \end{aligned}$$

(iv)
$$\begin{aligned} a^4 - 16b^4 &= (a^2)^2 - (4b^2)^2 \\ &= (a^2 + 4b^2)(a^2 - 4b^2) \\ &= (a^2 + 4b^2)(a + 2b)(a - 2b). \end{aligned}$$

EXERCISE XVIII.

1. In the identity $a^2 - b^2 = (a+b)(a-b)$ substitute

(i) $2x$ for a, 1 for b. (ii) $5x$ for a, $3y$ for b.

(iii) x^2 for a, y^2 for b. (iv) x^3 for a, y^3 for b.

(v) $(x+y)$ for a, $4z$ for b. (vi) x for a, $(y-z)$ for b.

Express the following as the product of factors :

2. $x^2 - 1$.
3. $49 - x^2$.
4. $100 - a^2b^2$.
5. $36a^2b^2 - 25c^2d^2$.
6. $81x^8 - a^2y^6$.
7. $4a^2x^4 - 9b^4y^2$.
8. $16x^6 - y^8$.
9. $121x^2 - 64a^4$.
10. $x^6 - 36$.
11. $25x^{10} - 9$.
12. $x^4 - 16y^{16}$.
13. $a^2b^2 - 49x^4$.
14. $(a+b)^2 - c^2$.
15. $a^2 - (b+c)^2$.
16. $(a-b)^2 - c^2$.
17. $a^2 - (b-c)^2$.
18. $(a-b)^2 - (x-y)^2$.
19. $(a+b)^2 - (x+y)^2$.
20. $9a^2 - (b+c)^2$.
21. $a^2 - 9(b+c)^2$.
22. $25a^2 - 4(b-c)^2$.
23. $36a^2 - 49(b-c^2)$.
24. $(x-y)^2 - 1$.
25. $(x+2y)^2 - 4z^2$.
26. $4z^2 - (x-2y)^2$.
27. $8x^2 - 2$.
28. $3x^3 - 27x$.
29. $8x^4 - 2y^2z^2$.
30. $25x^4y^3 - y^7$.
31. $144x^3y^2z - x^5z$.
32. $a^3b - ab^3$.
33. $a^5b^3 - a^3b^6$.
34. $52x^5 - 13x^3$.
35. $a^4 - b^4$.
36. $a^8 - b^8$.
37. $16a^4 - 81b^4$.
38. $a^8 - 625a^4b^4$.
39. $32x^5 - 2x$.
40. $x^7 - x^3$.
41. $a^5b - 81ab^5$.
42. $(2x+y-z)^2 - (x+2y-2z)^2$.

45. Difference of two Squares (continued).

Ex. 1. *Resolve into factors* $a^2-2ab+b^2-9c^2$.

$$\begin{aligned}a^2-2ab+b^2-9c^2 &= (a^2-2ab+b^2)-9c^2\\ &= (a-b)^2-(3c)^2\\ &= (a-b+3c)(a-b-3c).\end{aligned}$$

Ex. 2. *Resolve into factors* $a^2-b^2+6bc-9b^2$.

$$\begin{aligned}a^2-b^2+6bc-9b^2 &= a^2-(b^2-6bc+9b^2)\\ &= a^2-(b-3c)^2\\ &= \{a+(b-3c)\}\{a-(b+3c)\}\\ &= (a+b-3c)(a-b+3c).\end{aligned}$$

Ex. 3. *Resolve into factors* $(5x+3y)^2-(3x-5y)^2$.

$$\begin{aligned}&(5x+3y)^2-(3x-5y)^2\\ &= \{(5x+3y)+(3x-5y)\}\{(5x+3y)-(3x-5y)\}\\ &= (5x+3y+3x-5y)(5x+3y-3x+5y)\\ &= (8x-2y)(2x+8y)\\ &= 2\,.\,(4x-y)\,.\,2\,.\,(x+4y)\\ &= 4(4x-y)(x+4y).\end{aligned}$$

Ex. 4. *Factorize* (i) a^4+4b^4; (ii) $a^4+a^2b^2+b^4$.

$$\begin{aligned}\text{(i)}\quad a^4+4b^4 &= a^4+4a^2b^2+4b^4-4a^2b^2\\ &= (a^2+2b^2)^2-(2ab)^2\\ &= (a^2+2ab+2b^2)(a^2-2ab+2b^2).\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad a^4+a^2b^2+b^4 &= a^4+2a^2b^2+b^4-a^2b^2\\ &= (a^2+b^2)^2-(ab)^2\\ &= (a^2+ab+b^2)(a^2-ab+b^2).\end{aligned}$$

EXERCISE XIX.

Factorize the following expressions :

1. $x^2+2xy+y^2-16$.

2. $x^2-2x+1-4y^2$.

3. $x^2-4xy+4y^2-z^2$.

4. $x^2-10xy+25y^2-9z^2$.

5. $x^2+14x+49-9y^2$.

6. $x^2-12xy+36y^2-25z^2$.

7. $4x^2+12xy+9y^2-z^2$.

8. $9x^2-30xy+25y^2-4z^2$

9. $4x^2+28x+49-y^2$.

10. $x^2+y^2-z^2-2xy$.

11. $x^2+y^2-z^2+2xy$.

12. $4x^2+25y^2-z^2-20xy$.

13. $x^2-y^2+2yz-z^2$.

14. x^2-y^2-2y-1.

15. $4x^2-9y^2+6y-1$.

16. $9x^2-10yz-25y^2-z^2$.

17. $16x^2-9y^2+12yz-4z^2$.

18. $49x^2-9y^2-42y-49$.

19. $x^2-25y^2-9z^2+30yz$.

20. $a^2+b^2-c^2-d^2+2ab-2cd$.

21. $a^2-b^2+c^2-d^2-2ac+2bd$.

22. $(3x+2)^2-(2x-3)^2$.

23. $4(x+1)^2-(x-3)^2$.

24. $9(2x+3)^2-(5x-6)^2$.

25. $25(3x-2)^2-4(x-3)^2$.

26. $81(x-2)^2-4(x+2)^2$.

27. x^4+x^2+1.

28. $x^4+4x^2y^2+16y^4$.

29. x^4-7x^2+1.

30. $x^4-11x^2y^2+y^4$.

31. $x^4+3x^2y^2+4y^4$.

32. x^4+4.

33. x^4+64y^4.

Express the following as the product of as many factors as possible:

34. $(5x^2-x-1)^2-(x+6)^2$.

35. $(x^2+5x+7)^2-(2x+5)^2$.

36. $(2x^2+x+5)^2-(7x+1)^2$.

37. $(12x^2+22x+3)^2-(8x+9)^2$.

38. $(12x^2-31x+64)^2-(53x-56)^2$.

46. Sum or Difference of two Cubes. Any expression which is the sum or difference of two cubes may be factorized by means of the identities

$$\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2),}$$
$$\mathbf{a^3-b^3=(a-b)(a^2+ab+b^2).}$$

Ex. 1. *Factorize* (i) $8x^3+27y^3$; (ii) $250x^3-2$.

(i)
$$\begin{aligned} 8x^3+27y^3 &= (2x)^3+(3y)^3 \\ &= (2x+3y)\{(2x)^2-(2x)(3y)+(3y)^2\}^* \\ &= (2x+3y)(4x^2-6xy+9y^2). \end{aligned}$$

(ii)
$$\begin{aligned} 250x^3-2 &= 2(125x^3-1) \\ &= 2\{(5x)^3-1^3\} \\ &= 2(5x-1)(25x^2+5x+1) \end{aligned}$$

* In practice this step may be omitted.

Ex. 2. *Factorize* $8(x+y)^3-(2x-y)^3$.

The given expression

$$=\{2(x+y)-(2x-y)\}\{4(x+y)^2-2(x+y)(2x-y)+(2x-y)^2\}$$
$$=\{3y\}\{4(x^2+2xy+y^2)-2(2x^2+xy-y^2)+(4x^2-4xy+y^2)\}$$
$$=3y(4x^2+8xy+4y^2-4x^2-2xy+2y^2+4x^2-4xy+y^2)$$
$$=3y(4x^2+2xy+7y^2).$$

EXERCISE XX.

Factorize the following expressions :

1. x^3+1. **2.** x^3-1. **3.** $1+8x^3$.

4. $27x^3-1$. **5.** $27x^3-y^3$. **6.** x^3+64y^3.

7. $216x^3-y^3$. **8.** $27x^3-8y^3$. **9.** $343x^3+1$.

10. $x^3y^3+z^3$. **11.** x^3y^3-1000. **12.** $512+x^3$.

13. $x^3y^3-8z^3$. **14.** $343x^3-8y^3$. **15.** $8x^3+125y^3$.

16. $1+1000x^3$. **17.** $16x^4+2x$. **18.** x^4y-xy^4.

19. $x^5y^2+8x^2y^5$. **20.** $40x+625x^4$. **21.** $2x^3y^3z^3-16$.

22. x^6+27y^3. **23.** $9x^9+72x^6$. **24.** x^6+y^6.

25. x^6+1. **26.** x^6-1. **27.** x^6+64.

28. x^6-64. **29.** x^6+729y^6. **30.** x^6-729y^6.

31. $(2x+y)^3+(x+2y)^3$. **32.** $(5x+3y)^3-(x+y)^3$.

33. $(5x-3y)^3-(x-y)^3$. **34.** $27x^3-(3x-y)^3$.

For miscellaneous exercises on factors see pp. 188–190.

CHAPTER VII.

EQUATIONS AND PROBLEMS.

47. Simple Equations. An equation which, when reduced to its simplest form, contains no power of the unknown (x) higher than the first, is called a **simple equation.** (Thus, a simple equation does not contain x^2, x^3, or any higher power of x.)

Ex. 1. *Solve the equation* $4+3x=5x-8$.

The given equation is $4+3x=5x-8$.

Subtracting $3x$ from each side, we have

$$4=5x-8-3x;$$

$$\therefore\ 4=5x-3x-8. \quad (\textit{Commutative Law})$$

Adding 8 to each side,

$$4+8=5x-3x;$$

$$\therefore\ 12=2x;$$

$$\therefore\ 2x=12. \quad (\textit{Rule for Equalities})$$

Dividing each side by 2, $\quad x=6$.

In practice, the following **Rule** is used :

An equation is not altered (that is to say, it continues to be satisfied by the same value or values of the unknown), if any term is transposed from one side of the equation to the other, the sign of the term being changed from + to −, or from − to +.

This rule is merely a practical way of applying Rules 2 and 3 of Art. 24, which state that an equation is not altered if the same number is added to both sides, or if the same number is subtracted from both sides.

Thus, taking the equation of the last example,

$$4+3x=5x-8,$$

and transposing the terms $+3x$ and -8 each to the other side of the equation, and changing the signs of these terms, we have, as before,

$$4+8=5x-3x.$$

A bracket having the meaning assigned to it in Art. 9, an expression enclosed in a bracket is to be regarded as a *term*. Thus the equation

$$3x=2-(5x-6)$$

may be written in the form

$$3x+(5x-6)=2.$$

Ex. 2. *Solve the equation*

$$3(2x+3)(3x+2)-2(3x+1)^2=43.$$

Here the multiplications should be performed mentally; the work should be arranged as follows,

$$3(6x^2+13x+6)-2(9x^2+6x+1)=43;$$
$$\therefore\ 18x^2+39x+18-18x^2-12x-2=43;$$
$$\therefore\ 39x-12x=43+2-18;$$
$$\therefore\ 27x=27;$$
$$\therefore\ x=1.$$

48. Equations with Literal Coefficients. Consider the equation

$$(x+a)(x-b)=x^2+a^2+b^2-3ab,\ldots\ldots\ldots\ldots(\alpha)$$

where a and b are supposed to stand for known numbers. Assuming that x stands for a number whose value can be found in terms of a and b from the given equation, and expanding the left hand side of (α), we have

$$x^2+x(a-b)-ab=x^2+a^2+b^2-3ab;$$
$$\therefore\ x(a-b)=a^2-2ab+b^2;$$
$$\therefore\ x(a-b)=(a-b)^2.\ \ldots\ldots\ldots\ldots\ldots(\beta)$$

Now *if a is greater than b*, $a-b$ stands for a known number, and we may divide each side of (β) by $a-b$, thus

$$x=a-b.$$

Note that *if $a=b$, the equation (α) reduces to an identity and is satisfied for all values of x.*

EXERCISE XXI.

Solve the equations:

1. $5x+56=2x+71$.
2. $7x-38=2x+12$.
3. $10x-49=49-4x$.
4. $7x-32-(5x-16)=8$.
5. $3(x+14)+7(x+9)=5(3x+2)$.
6. $(3x+6)-(19-6x)=(5x+61)-(27x+12)$.
7. $5(x-2)-6(11-x)=3(2x-11)-2(x-3)$.
8. $(x+1)(x+2)=(x-1)^2+11$.
9. $(x+3)^2+(x+5)^2=2(x+2)^2+34$.
10. $3(x+2)^2+2(x+3)^2=5(x-1)^2+7(6x-1)$.
11. $(4x-1)(6x+17)=(3x-2)(8x+3)+58$.
12. $(4x+1)(3x-2)-(6x-1)(2x-3)=25$.
13. $(x-4)(2x-11)-(x-6)(2x-13)=14$.
14. $4(2x+3)(3x+1)-12(x+1)^2=(4x+5)(3x+1)$.
15. $(4x+7)(11-5x)-9=11-5(2x-3)^2$.
16. $(2x+3)(3x+2)+(4x-5)(x+1)=(5x-1)(2x+3)$.
17. $(x-2)(x-3)+5(x-2)(x-4)-6(x-3)(x-4)=16$.
18. $(x-1)(x-4)-(x-2)(x-5)=(x+1)^2-x(x-3)-37$.
19. $5(2x-1)(3x+2)-2(7x+4)(x-2)=50+2x+(4x+1)^2$.
20. $(x+4)(x+5)(x+6)=(x+3)(x+4)(x+8)+30$.
21. $(x-3)^3+101=(x-4)^3+3(x+1)^2$.
22. $(x-2)^3+(x-3)^3+(x-4)^3=3(x-2)(x-3)(x-4)+18$.

Find the value of x in terms of a and b which satisfies each of the following equations:

23. $4(x-a)=3(x+b)$.
24. $3(x-2a)+4(x-a)=5x$.
25. $x(a+b-c)-x(a-b-c)=4b^2$.
26. $(x-b)(a+b)=(x+b)(a-b)$.
27. $x(a+2)-6=a(x+6)-2$.
28. $a(a-x)=b(b+x)$.
29. $a(x-a)=b(x-b)$.
30. $a(x-a)-b(x-2a)=b^2$.
31. $x(a-x)-a^2=x(b-x)-b^2$.
32. $abx=a^2(a-x)-b^2(b+x)$.
33. $a(a-x)=2ab-b(x+b)$.
34. $a(x-a)-b(x-b)=a-b$.
35. $(a+b)(x-b)-(a-b)(x-a)=a^2+b^2$.
36. $(a-3)(x-a)=2(3a-x-2)$.
37. $4a^2+(2a-x)(2b-x)=4b^2+(2a+x)(2b+x)$.

38. $ax+b^3=bx+a^3$.

39. $ax-a^3=27-3x$.

40. $a(x-a)+b(x-b)=2ab$.

41. $a(a+x)+b(b-x)=2ab$.

49. The Decimal System of Numeration. In accordance with the notation of Arithmetic, 237 stands for

$$(100\times 2)+(10\times 3)+7,$$

and 2, 3, 7 are called the **digits** of the number 237. In the same way, if the digits of a number, taken in order from left to right, are x, y, z, the number is denoted by

$$100x+10y+z.$$

Ex. 1. *Find a number of two digits which is such that if 54 is added to the number, the order of the digits is reversed. How many such numbers are there?*

Let $10x+y$ be such a number. Then, by hypothesis,

$$10x+y+54=10y+x,$$
$$\therefore\ 54=9(y-x),$$
$$\therefore\ y=x+6.$$

Now the least value which x can have is 1, and the corresponding value of y is 7; if $x=2$, $y=8$; if $x=3$, $y=9$, and 9 is the greatest value which y can have. Hence there are three numbers of the kind mentioned, namely 17, 28 and 39.

50. Problems leading to Simple Equations.

Ex. 1. *A and B have £99 between them, and A wins from B twice as much as A had originally; A has then twice as much as B. What sum had each originally?*

If A had £x originally, then B had £$(99-x)$. After winning £$(2x)$ from B, A has £$(3x)$ and B has £$(99-3x)$; and since A has now twice as much as B,

$$\therefore\ 3x=2(99-3x);$$
$$\therefore\ 3x=198-6x;$$
$$\therefore\ 9x=198,$$
$$\therefore\ x=22.$$

Thus A had £22 and B had £77.

Ex. 2. *Find the nth term of the arithmetical progression*

$$5,\ 11,\ 17,\ \ldots.$$

If the nth term of this series is 125, what is the value of n?

Here the common difference is 6,

hence the nth term $= 5 + (n-1)6 = 6n - 1$. [Art. 10]

Also, if $6n - 1 = 125$, then $6n = 126$, $\therefore\ n = 21$.

EXERCISE XXII.

1. When A travels alone, he spends x shillings per day; when B travels alone, he spends y shillings per day. When A and B travel together, it costs each z shillings per day. How much do A and B save between them by travelling together for n days.

2. (i) If in question 1 the total amount saved by the two is s shillings, what is the formula for s? (ii) If $x = 16$, $y = 15$, $z = 12$ and the total amount saved is £7, for how many days did A and B travel together?

3. A father is forty years older than his son; seven years ago he was three times as old as his son: find their present ages.

4. Find two consecutive numbers such that the difference of their squares is 49.

5. A is twice as old as B; twenty-two years ago he was three times as old; what is A's present age?

6. A sum of £5 is divided between two men, so that one receives eight times as many pence as the other receives shillings. What sum does each receive?

7. A man has five sons whose united ages equal his own. In twelve years' time the united ages of the sons will be double that of the father. What is the father's age now?

8. A father is 30 years older than his son; in 20 years' time his age will be double that of the son. Find the present age of the son.

9. Find two numbers whose sum is 23, such that the difference between their squares is 161.

10. A number is formed with two digits whose sum is 9; if 63 is subtracted from the number the order of the digits is reversed. Find the number.

11. A and B have between them £400; A receives a legacy of £350 and then he has twice as much as B. How much had each at first?

12. In five years from now a man will be three times as old as his son, and in seventeen years from now he will be twice as old: find their present ages.

13. A has a certain number of shillings, and B has the same number of pennies. A gives B three shillings in exchange for their value in pennies, and it is now found that A has twice as many coins as B. How many coins had each at first?

14. A purse contains £9. 10*s.* in sovereigns and half crowns, and the total number of coins is 20. Find how many sovereigns there are in the purse.

15. Twenty years hence I shall be seven times as old as I was 28 years ago; how old am I now?

16. A bag contains 129 coins, some of which are sovereigns and the rest shillings; they amount altogether in value to £41. 12*s.* How many coins of each kind are there?

17. Three times A's age exceeds twice B's age by 35 years, and in 20 years the sum of their ages will be double what it is now. Find their ages.

18. A debt, which might have been paid with $5x$ half sovereigns and $6x$ half crowns, was paid out of a £10 note, and the change was found to be equal to x half sovereigns and $10x$ half crowns. Find x and the amount of the debt.

19. Find three consecutive numbers such that three times the greatest shall be equal to twice the sum of the other two.

20. A bill of six guineas is paid in sovereigns, half crowns and shillings. There are twenty-six coins altogether, and there are four times as many half crowns as sovereigns. Find the number of each coin.

21. Divide £400 between A, B and C, so that C's share may be £70 more than twice A's and £40 less than three times B's.

22. If 11 is taken from a certain integer we get the square of a whole number, and if 24 is added to the same integer we get the square of the next greater number. Find the integer.

23. I bought some engravings at 35*s.* each and some books at 16*s.* each. The total cost was £16. 15*s.* and the number of books was 5 more than the number of engravings. How many were there of each?

24. A square court is paved with flags, each a foot square. If the length of the court were increased by 5 feet and its breadth by 2 feet, 164 more flags would be required. What is the length of the court?

25. Two towns are 60 miles apart; coaches ply between them in opposite directions, starting at the same time, one going 2 miles an hour faster than the other; they meet each other after 3 hours: find where they meet.

26. The prices of seats in the stalls, pit and gallery of a theatre are respectively seven shillings and sixpence, half-a-crown and one shilling. The pit can hold three times, and the gallery twice, as many people as the stalls. The receipts are £87. 6*s.* when all the seats in the pit and gallery are occupied and all but twelve in the stalls; how many people are present?

27. There are two numbers of two digits such that each number is 14 times its left-hand digit: find the numbers.

28. There are four numbers of two digits such that each number is four times the sum of its digits. Find the numbers.

29. Find a number of two digits such that if 36 is added to the number the order of its digits is reversed. Write down all the numbers of this kind.

30. A number has three digits x, y, z. Prove that

(i) If the number is diminished by the sum of its digits the remainder is divisible by 9.

(ii) If $y=z+x$, then the number is divisible by 11.

(iii) Of what rules in Arithmetic are these properties of numbers instances?

31. A number N has four digits x, y, z, w. Prove that

(i) If $x+y+z+w$ is divisible by 9, then N is divisible by 9.

(ii) $N-(w-z+y-x)$ is divisible by 11.

(iii) If $w-z+y-x$ is divisible by 11, then N is divisible by 11.

State the rule in Arithmetic of which (iii) is a particular case.

32. Prove that the product of any two consecutive odd numbers increased by unity is the square of an even number.

33. Prove that the product of any two consecutive even numbers increased by unity is the square of an odd number.

34. Prove that four times the product of two consecutive numbers increased by unity is the square of an odd number.

35. Prove that the difference of the squares of any two consecutive odd numbers is divisible by 8.

36. Find the nth term of the arithmetical progression 3, 8, 13, ... If the nth term is 203, what is n?

37. A clerk receives £100 for his first year's service, and his salary is to rise £3 a year. During his nth and $(n+1)$th years he receives altogether £215. What is the value of n?

38. The ath term of the progression 2, $(2+x)$, $(2+2x)$, ... is $(7a-5)$. Find the value of x, and write down the 11th term.

39. The difference of the ath and bth terms of an arithmetic progression is $5(a-b)$. Find the common difference of the progression.

51. Two or more Unknowns connected by one Equation. If the numbers 1, 2, 3, 4, ... are substituted in succession for x in the equation $y=2x+3$, the corresponding values of y are 5, 7, 9, 11, ... Thus x and y may have any of the pairs of values (1, 5), (2, 7), (3, 9), ..., where the first number in a "pair" is a possible value of x and the second number in the pair is the corresponding value of y.

DEF. If an equation contains more than one unknown, any *set of values* of the unknowns which satisfies the equation is called a **solution**.

Thus, to say that (2, 7) is a solution of $y=2x+3$ is to say that this equation is satisfied if $x=2$ and $y=7$.

Again, if x and y denote *natural numbers*,

(1) the solutions of $x=5-2y$ are (3, 1), (1, 2);

(2) the solutions of $xy=6$ are (6, 1), (3, 2), (2, 3), (1, 6).

If x and y stand for two natural numbers which are connected by one equation, it appears from the preceding that the equation has in general several solutions; the example at the end of this article shows that the number of solutions may be unlimited.

If the restriction that x and y are to stand for *natural* numbers is removed, and if these letters may denote numbers of the various classes *not yet discussed*, it will be found that *every* single equation in x and y has an unlimited number of solutions.

It must be observed that when two unknowns, denoted by x and y, are considered, the equation $x=3$ means that x stands for 3 and that y may have any value. In this sense, the equation $x=3$ has an unlimited number of solutions.

Ex. *Show that if n stands for any number, the equation $4y=5x+3$ is satisfied when $x=4n+1$, $y=5n+2$.*

If $x=4n+1$ and $y=5n+2$, we have

$$4y=4(5n+2) \quad =20n+8$$

$$\text{and}\quad 5x+3=5(4n+1)+3=20n+8\,;$$

$$\therefore\ 4y=5x+3.$$

Hence, the given equation is satisfied when

$$x=4n+1,\ y=5n+2,$$

whatever value n may have. The number of solutions is therefore unlimited.

52. Simultaneous Equations. We proceed to enquire if it is possible to find a pair of values of x and y which will satisfy two given equations in which these letters occur.

Def. Equations containing two or more unknowns are called **simultaneous equations** if each unknown letter is supposed to stand for the same number in all the equations. Any set of values of the unknowns which satisfies all the equations is called a **solution** of the equations.

Ex. *Search for a solution of the equations*

$$x+2y=11, \quad \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$

$$x-2y=3. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

We assume that two numbers exist such that if these are substituted for x and y in (α) and (β), the equations are satisfied, and we assume that x and y stand for two such numbers.

From equation (β), we have

$$x=2y+3.$$

Thus $(2y+3)$ stands for the same number as x, and may therefore be substituted for x in (α), so that

$$(2y+3)+2y=11, \tag{γ}$$

$$4y+3=11,$$

$$4y=8,$$

$$y=2. \tag{δ}$$

Thus y stands for 2, and we may substitute 2 for y in either of the given equations. Putting 2 for y in (β), we have

$$x-2\,.\,2=3, \tag{ϵ}$$

$$x=7. \tag{ζ}$$

It has now been shown that if there is a solution, the solution is $(7, 2)$; it remains to prove that $(7, 2)$ is a solution.

When $x=7$ and $y=2$, we have

$$x+2y=7+2\,.\,2=7+4=11$$

$$\text{and}\quad x-2y=7-2\,.\,2=7-4=3.$$

Hence the given equations are satisfied and $(7, 2)$ is a solution, and is the only solution.

53. Reversible Operations. The method employed in the last example is called the **method of substitution**. It will now be shown that every step in the process which leads from equations (α), (β) to equations (δ), (ζ) is reversible.

Starting with equations (α), (β), it follows from (β) that x and $(2y+3)$ stand for the same number. Hence any solution of (α) and (β) is also a solution of (γ) and (β). This justifies the first step in the solution, which is to replace equations (α), (β) by equations (γ), (β).

That this step may be reversible, it must be shown that any solution of (γ) and (β) is also a solution of (α) and (β). This is the case: for by (β), x and $(2y+3)$ stand for the same number, and therefore equations (α), (β) can be derived from (γ), (β).

The step from (γ) to (δ) is reversible by the Rules for Equalities.

The next step is to replace the equations $y=2$ and $x-2y=3$ by the equations $y=2$ and $x-2\,.\,2=3$. This step is reversible, for y stands for 2.

The step from (ϵ) to (ζ) is reversible by the Rules for Equalities. Thus all the steps in the solution are reversible, and $(7, 2)$ is the solution.

It will now be understood that

(i) **substitution is a reversible operation ;**

(ii) **when a solution of a set of simultaneous equations is obtained by a series of reversible steps, no further verification is required.**

Ex. 1. *Solve the equations*

$$2x=3y, \quad \text{.............................}(\alpha)$$

$$5x+2y=19. \quad \text{.............................}(\beta)$$

Multiplying each side of (α) by 5,

$$10x=15y. \quad \text{.............................}(\gamma)$$

Multiplying each side of (β) by 2,

$$10x+4y=38. \quad \text{.............................}(\delta)$$

Substituting $15y$ for $10x$ in (δ),

$$15y+4y=38\,;$$

$$\therefore\ 19y=38\,;$$

$$\therefore\ y=2.$$

Substituting 2 for y in (α),

$$2x=6\,;$$

$$\therefore\ x=3.$$

The solution is therefore (3, 2).

54. Elimination. To eliminate an unknown x between two given equations, is to derive from these an equation which does not contain x.

Thus, $(2y+3)+2y=11$ is the result of eliminating x between the equations $x+2y=11$ and $x-2y=3$.

A modification of the method in Art. 52 is to compare the values of x in terms of y (or the values of y in terms of x) derived from the given equations.

Thus, from equation (α) of Art. 52,

$$x=11-2y,$$

and from (β), $\qquad x=2y+3.$

Equating these values of x,

$$2y+3=11-2y,$$

and the solution proceeds as before.

EXERCISE XXIII.

Solve the equations :

1. $x=2y$, $x+y=21$.

2. $x=5y$, $x-y=8$.

3. $x=3y$, $x+5y=16$.

4. $x=4y$, $2x-7y=3$.

5. $y=4x$, $9x-2y=2$.

6. $2x=5y$, $3x-7y=2$.

7. $3y=4x$, $8x-5y=4$.

8. $2y=7x$, $5x+y=51$.

9. $3x=5y$, $2x+3y=38$.

10. $y=3x+2$, $2x+3y=28$.

11. $x=5y-4$, $10y-3x=2$.

12. $x=2y+3$, $7y-2x=3$.

13. (i) What are the least integral values of x and y (that is the least values of x and y which are whole numbers) which satisfy the equation $2x=3y$?

(ii) What are the least integral values of x, y and z which satisfy the equations $2x=3y$ and $x+y=z$?

14. If $3x=5y$ and $x+2y=z$:

(i) Find the equation connecting y and z;

(ii) Find the equation connecting x and z;

(iii) Find the least integral values of x, y and z which satisfy all these equations.

15. Eliminate y between the equations

$$y=2x+3, \quad 3y-2x=a.$$

Also find the values of x and y when $a=29$.

16. If x and y stand for whole numbers which satisfy the equation $x=3y+2$, find the values of x which lie between 40 and 49.

17. Eliminate y between the equations

$$y=3x-4, \quad 5x-2y=a.$$

Also find the values of x and y when $a=3$.

18. Find x and y in terms of a and b from the equations

$$3y=a+2x, \quad 3x-4y=b.$$

Also find the values of a and b if these equations are satisfied when $x=5$ and $y=6$.

19. If the three expressions $4y$, $2x+3y$, $x+y+25$ have equal values, find the value of each expression.

55. Method by Addition and Subtraction. A third method of solution, called the method by addition or subtraction, is probably the method which is most generally useful.

If the given equations are

$$2x+3y=20, \quad \ldots\ldots(\alpha)$$
$$3x-5y=11. \quad \ldots\ldots(\beta)$$

Multiplying each side of (α) by 5, and each side of (β) by 3,

$$5(2x+3y)=100,$$
$$3(3x-5y)=33.$$

By addition $\quad 5(2x+3y)+3(3x-5y)=133, \quad \ldots\ldots(\gamma)$

$$19x=133,$$
$$x=7. \quad \ldots\ldots(\delta)$$

Substituting 7 for x in (α),

$$2.7+3y=20,$$
$$3y=6,$$
$$y=2.$$

In this process, we replace equations (α) and (β) by equations (α) and (γ). Now (α) and (β) can be derived from (α) and (γ), so that this part of the process is reversible. As in Art. 23, it can be shown that the rest of the process is reversible; hence (7, 2) is the solution.

It will be seen that the search for a solution depends on the application of Theorems 7 and 8 of Art. 20; whilst the reversibility of the steps depends on the truth of the converses of these theorems as stated in Art. 21.

56. Practical Arrangement. In solving the equations of the last article, we may commence by eliminating x. In practice the solution of such equations is written as follows:

Ex. 1. *Solve the equations*

$$2x+3y=20, \quad \ldots\ldots(\alpha)$$
$$3x-5y=11 \quad \ldots\ldots(\beta)$$

Multiplying each side of (α) by 3, and each side of (β) by 2,

$$6x+9y=60, \quad \ldots\ldots(\gamma)$$
$$6x-10y=22. \quad \ldots\ldots(\delta)$$

From (γ) and (δ), by subtraction,

$$19y = 38,$$
$$y = 2.$$

Substituting 2 for y in either (α) or (β), etc.

Ex. 2. *Solve the equations*

$$51x + 23y = 199, \quad\ldots\ldots\ldots(\alpha)$$
$$85x - 32y = 191. \quad\ldots\ldots\ldots(\beta)$$

Since $51 = 17 \times 3$ and $85 = 17 \times 5$, $\therefore\ 51 \times 5 = 85 \times 3$. Thus, to eliminate x, we multiply both sides of (α) by 5 and both sides of (β) by 3 and subtract. The solution proceeds as before.

Ex. 3. *Find the values of x and y in terms of a from the equations*

$$2ax - y = 2a^3, \quad\ldots\ldots\ldots(\alpha)$$
$$ay - x = a^2 - 1. \quad\ldots\ldots\ldots(\beta)$$

Multiplying each side of (α) by a,

$$2a^2x - ay = 2a^4. \quad\ldots\ldots\ldots(\gamma)$$

From (β) and (γ), by addition,

$$2a^2x - x = 2a^4 + a^2 - 1,$$
$$\therefore\ x(2a^2 - 1) = 2a^4 + a^2 - 1 = (2a^2 - 1)(a^2 + 1).$$

Dividing each side by $2a^2 - 1$,

$$x = a^2 + 1.$$

Substituting $(a^2 + 1)$ for x in (β),

$$ay - (a^2 + 1) = a^2 - 1,$$
$$\therefore\ ay = (a^2 - 1) + (a^2 + 1) = 2a^2,$$
$$\therefore\ y = 2a,$$

Hence $x = a^2 + 1$, $y = 2a$.

57. Equivalent sets of Equations. To say that $a = b = c$ is to say that a, b, c stand for the same number: this statement is therefore equivalent to any one of the three pairs of equations (1) $b = c$, $c = a$; (2) $c = a$, $a = b$; (3) $a = b$, $b = c$.

Ex. 1. *Solve the equations*

$$2(6x-7y)=5(x-2)-(2y+3)=3x-11.$$

The given equations are equivalent to (that is to say they have the same solutions as)

$$2(6x-7y)=3x-11 \quad\ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$

and $$5(x-2)-(2y+3)=3x-11. \quad\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

From (α), $$12x-14y=3x-11,$$
$$\therefore\ 11=14y+3x-12x,$$
$$\therefore\ 14y-9x=11. \quad\ldots\ldots\ldots\ldots\ldots\ldots(\gamma)$$

From (β), $$5x-10-2y-3=3x-11,$$
$$\therefore\ 5x-3x-2y=10+3-11,$$
$$\therefore\ 2x-2y=2,$$
$$\therefore\ x-y=1. \quad\ldots\ldots\ldots\ldots\ldots\ldots(\delta)$$

Equations (α), (β) are therefore equivalent to equations (γ), (δ). Multiplying both sides of (δ) by 14 and adding to (γ), we find $x=5$; substituting 5 for x in (δ), we have $y=4$.

EXERCISE XXIV.

Solve the following simultaneous equations, and in questions 1–4 verify the results by actual substitution:

1. $x+y=112$, $x-y=48$.

2. $x+2y=23$, $2y-x=5$.

3. $5x-7y=22$, $11x-7y=82$.

4. $7y-6x=2$, $6x+8y=118$.

5. $2x+7y=25$, $3x+9y=33$.

6. $5x+3y=47$, $3x-5y=1$.

7. $7x-8y=7$, $5y-4x=11$.

8. $19x-3y=178$, $17x+24y=266$.

9. $4x=3y-3$, $2y=3x-1$.

10. $60x-17y=285$, $75x-19y=390$.

11. $37x+57y=541$, $43x-95y=145$.

12. $58x-23y=244$, $87x-43y=349$.

13. $236x-51y=319$, $295x\quad 67y=389$.

14. If $3x+4y=96$ and $6x-7y=12$, prove that $3x=4y$.

15. If $9x-8y=65$ and $2x+3y=110$, prove that $4x=5y$.

16. Explain why no solutions exist in the case of the equations

$$x+2y=3,\ 4(x+2y)=8.$$

17. Find all the integral solutions of the equations

$$x+3y=13,\ 2(x+3y)=26$$

(that is to say, find all the pairs of natural numbers which are solutions).

18. $7x-5y+6=13y-5x-6,$
$2y-x=2.$

19. $5(x-3)-2(y+1)=2(2x-3y),$
$x=y+2.$

20. $6x-7y+4=3x+2y-11=4x-5y+6.$

21. $5(x+y)-13(x-y)+7=2x+3y+2=3(2x-y)+2(2y-x)-6.$

Find x and y in terms of the other letters from the following equations :

22. $x+y=2a,$
$x-y=2b.$

23. $3x+4y=7a+11b,$
$y-x=b.$

24. $5x-6y=11b-a,$
$x+y=2a.$

25. $x+y=3(a+b-c),$
$2x-y=3a.$

26. $ax+y=a^2,$
$y-x=1.$

27. $ax+by=(a+b)^2,$
$ax-by=a^2-b^2.$

28. $a^2x+b^2y=ab(a+b),$
$bx+ay=a^2+b^2.$

29. $a^2x+y=a^3,$
$(a+1)x+1=y.$

30. $ax=by,$
$(a+b)y-(a-b)x=a^2+b^2.$

31. $ax-y=a(a+1),$
$x(a-1)=y-1.$

32. $ax-by=a^2,$
$(b-a)x+ay=b^2.$

33. $x+y=2(bx-ay)=2(a+b).$

34. $a^3x-y=x+a^3y=a^6+1.$

35. $(a+b)x-ay=3ab+b^2,$
$ax-(a-b)y=3ab-b^2.$

36. If $ax+by=c$ and $x-y=1$, prove that $x(c-a)=y(b+c)$.

37. If $y=ax+b$ and $x=py-q$, prove that $y(q-bp)=x(aq-b)$.

58. Problems leading to Simultaneous Equations.

Ex. 1. *A man spent £15 in buying two kinds of tea, one kind costing 2s. 6d. per lb., and the other 2s. per lb. Having mixed the two kinds together, he sold the mixture at 3s. per lb., and gained £4. 10s. by the transaction. How much of each kind did he buy?*

Suppose that he bought x lbs. of the first kind, and y lbs. of the second kind;

then 1 lb. of the first kind cost 5 sixpences,

$\therefore$ x lbs. " " $5x$ sixpences;

and 1 lb. of the second kind cost 4 sixpences,

$\therefore$ y lbs. " " $4y$ sixpences;

$\therefore$ total cost of tea $=(5x+4y)$ sixpences.

Now the total cost $=$ £15 $=600$ sixpences;

$$\therefore\ 5x+4y=600. \qquad (\alpha)$$

Again, since 1 lb. of the mixture sold for 6 sixpences;

$\therefore$ $(x+y)$ lbs. " " $6(x+y)$ sixpences.

Now the selling price of mixture $=$ £19. 10s. $=780$ sixpences;

$$\therefore\ 6(x+y)=780;$$

$$\therefore\ x+y=130. \qquad (\beta)$$

Solving the equations (α), (β), we find that

$$x=80,\ y=50.$$

Thus he bought 80 lbs. of the first kind, and 50 lbs. of the second kind.

Ex. 2. *The perimeter of a rectangle is 26 inches; if the length of the rectangle is increased by 2 inches and the breadth diminished by 1 inch, the area is unaltered. Find the length and breadth.*

If the length is x inches and the breadth y inches, the perimeter is $2(x+y)$ inches.

$$2(x+y)=26,$$

$$x+y=13. \qquad (\alpha)$$

Again, the area of the rectangle is xy square inches, and when the length is increased by 2 inches and the breadth is diminished by 1 inch, the area is $(x+2)(y-1)$ square inches.

Hence, since the area is unaltered,

$$(x+2)(y-1)=xy,$$

$$\therefore\ xy+2y-x-2=xy,$$

$$\therefore\ 2y-x=2. \qquad (\beta)$$

From (α) and (β), by addition,

$$3y = 15,$$
$$\therefore\ y = 5$$

Also from (α), $\quad x + 5 = 13,$

$$\therefore\ x = 8.$$

The length is therefore 8 inches and the breadth 5 inches.

Ex. 3. *The 8th term of an arithmetical progression is 38 and the 13th term is 63; find the first term and the 10th term.*

Let the progression be

$$a,\ (a+d),\ (a+2d). \ldots\ldots\ldots\ldots(\text{See Art. } 10)$$

The 8th term of this series is $a + 7d$ and the 13th term is $a + 12d$;

$$\therefore\ a + 12d = 63, \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$
$$\text{and}\quad a + 7d = 38; \ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$
$$\therefore\ (a + 12d) - (a + 7d) = 25;$$
$$\therefore\ 5d = 25;$$
$$\therefore\ d = 5.$$

Substituting this value for d in (β),

$$a + 7 \cdot 5 = 38;$$
$$\therefore\ \text{first term } a = 3$$
$$\text{and 10th term} = a + 9d = 3 + 9 \cdot 5 = 48.$$

EXERCISE XXV.

In each of the examples 1-4, two terms of an arithmetical progression are given, and it is required to find some other term.

1. 20th term $= 58$, 30th term $= 88$; find the 10th term.
2. 11th term $= 72$, 19th term $= 128$; find the 24th term.
3. 2nd term $= a + 1$, 5th term $= a + 19$; find the 1st term.
4. 7th term $= x + 17y$, 11th term $= x + 29y$; find the 4th term.
5. Of two numbers twice the excess of the first over the second exceeds the sum of the numbers by 11, and three times the first exceeds the second by 49; find the numbers.

6. The cost of 7 ducks and 9 chickens is 55*s.*, and the cost of 13 ducks and 5 chickens is 67*s.* What is the cost of a chicken?

7. A bag contains only crowns and shillings, the number of coins being 120, and their value £11. 12*s.*; how many were there of each?

8. *A* has 13*s.* 4*d.* more than *B*. Find how much money each person has if 3 times *A*'s money added to twice *B*'s money just makes up £5.

9. Of a certain room, 3 times the breadth is equal to twice the length. If the breadth had been 3 ft. more and the length 3 ft. less, they would have been equal. Find the length and breadth of the room.

10. If 5 lbs. of tea and 7 lbs. of coffee cost £1. 3*s.* 4*d.*, and 8 lbs. of tea and 3 lbs. of coffee cost £1. 3*s.* 8*d.*, find the cost of tea and coffee per lb.

11. A number of two digits is 7 times the sum of its digits, and if the order of the digits is reversed, the number is diminished by 9. Find the number.

12. A certain number, consisting of two digits, exceeds four times the sum of its digits by 3; if the number is increased by 18, the result is the same as if the number formed by reversing the digits were diminished by 18. What is the number?

13. In 10 years *A* will be twice as old as *B* was 10 years ago. *A* is 9 years older than *B*. Find their ages now.

14. A man bought 86 yards of cloth, part of it at 3*s.* 6*d.* a yard and the rest at 4*s.* 6*d.* a yard. By selling the whole at 4*s.* 2*d.* a yard, he gained 7*s.* 4*d.* How many yards were there at each price?

15. A man bought for £9. 6*s.* two kinds of tea, one of which cost 2*s.* 4*d.* per lb. and the other 1*s.* 10*d.* Having mixed them he sold the mixture for 2*s.* 2*d.* per lb. and gained 9*s.* by the transaction; how many lbs. of each sort did he buy?

16. *A* and *B* have 51 coins between them, which are all shillings or pennies. *A* has 4 times as many shillings as pennies, and *B* has 3 times as many shillings as pennies. *A* has 3*s.* 11*d.* more than twice what *B* has. Find what money each had.

17. Find the price of a package of cloth and of a package of silk if five such packages of cloth and eight such of silk together cost £251, while seven such packages of cloth together cost £8 more than nine such of silk.

18. A man's estate, value £15,000, is to be divided between his wife, son and three daughters, so that the widow shall have £500 less than the four children together, and the son twice as much as each daughter. Find the share of each.

19. A man agreed to sell 2500 tons of cement at 20*s.* a ton. That which he had cost him 15*s.* a ton. Not having enough, he had to buy cement at 22*s.* a ton, to fulfil his agreement. He lost £5. How many tons had he to buy?

20. During a tour, A and B pay respectively £a and £b towards the common expenses. In order to share the expense equally A pays £x to B. Write down an equation connecting x, a, b. If the total expense of the tour is £66, and if A pays £10 to B, what are the values of a and b?

21. During a tour A and B spend respectively £a and £b. These sums cover the expense of the tour and also include £1 spent by A and £5 spent by B on personal matters. In order to share the common expense equally, A pays £x to B. What is the equation connecting x, a, b? If A's total expenses are £18 and if $x=2a$, find the value of b.

22. A farm consists of arable land let at 20*s.* per acre and pasture let at 30*s.* per acre, the total rent being £350; when the rent of the arable land is reduced by 5*s.* per acre, and that of the pasture by 8*s.* per acre, the total rent is reduced by £90. Find how many acres are in the farm.

23. Three times A's age is equal to four times B's age: in a years, four times A's age will be equal to five times B's age, and in b years five times A's age will be equal to six times B's age. Prove that $b=2a$.

24. A train travelled a certain distance at a uniform rate of speed. Had the speed been 6 miles an hour more, the journey would have taken 4 hours less; and, had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance travelled.

25. Two cyclists who have started from two towns 7 miles apart, find, when they meet, that each of their wheels has made 2880 complete revolutions. Find the circumference of each wheel in inches, it being observed that 18 revolutions of the smaller wheel cover 11 feet more ground than 12 revolutions of the larger.

26. A page of print contains 1300 words if in large type and 1850 if in small type. If an article of 37250 words occupies exactly 24 pages, how many pages are in small type and how many in large type?

27. Seven of *A*'s equal daily journeys to town cover altogether a distance 11 miles more than do nine of *B*'s. But *A*'s daily journey will soon be 13 miles longer than at present, and then two of *A*'s journeys will in distance be equal to five of *B*'s. Find the number of miles in a journey of *A* and in a journey of *B*.

28. A person having in his possession a certain sum consisting of pounds and shillings observes that, if the pounds were turned into shillings and the shillings into pounds, he would gain £2. 17*s.*; but that, if the pounds were turned into florins and the shillings into half-sovereigns, he would gain but nine shillings. Find the sum he had.

29. There are two samples of tea, one of them weighing 7 lbs. more than the other; if they are mixed, and the mixture is sold at 1*s.* 10*d.* per lb., the value realised will be the same as it would be if they were sold separately, the heavier sample at 1*s.* 6*d.* per lb. and the other at 2*s.* 4*d.* per lb. What is the weight of each sample?

30. Two boys, *A* and *B*, have money consisting of shillings and pennies. *A* has twice as many pennies as shillings; *B*, who has fivepence more than *A*, has three times as many pennies as shillings; together, twice the number of pennies they have is equal to five times the number of shillings they have. How much has each?

31. A certain sum of money is to be divided equally among a certain number of people. If the number of people had been greater by ten, the share of each would have been smaller by 9*s.* 2*d.*; and, if the number of people had been fewer by ten, the share

of each would have been larger by 13*s*. 4*d*. What was the sum of money to be divided?

32. A tourist travels for 5 hours by a train (X), then onward for 4 hours by a train (Y), and thus passes over 244 miles. He might have done the same distance by travelling for 1 hour by a train going 3 miles an hour faster than X, then onward for 11 hours by a train going 2 miles an hour slower than Y. Find in miles per hour the rates of the trains X and Y.

33. A says to B, "I am twice as old as you were when I was as old as you are; when you are as old as I am, I shall then be 7 years younger than twice your present age." Find the ages of A and B.

34. A room is such that, if it had been 5 feet longer and 3 feet narrower, its area would have been unaltered, and such that, if it had been 5 feet wider, it would have been a square. Find the length and breadth.

35. The garrison of a town has provisions for a certain number of days, and it is known that if a thousand men were to leave the garrison the provisions would last two days longer, but that if, instead, a thousand more men were to join it the provisions would last one day less. Find the number of the garrison.

[Observe that the number of men multiplied by the number of days must be the same in each case.]

36. Given $xy=(x+2)(y-4)=24$, where x and y stand for natural numbers:

(i) Prove that $y=2(x+1)$.

(ii) Find the sets of integral solutions of $xy=24$.

(iii) Which of these sets satisfy $y=2(x+1)$?

37. Find two natural numbers, x and y, which are connected by the equation $xy=(x+9)(y-3)=30$.

38. A room 432 square feet in area is to be covered, all except a margin of 2 feet in width all round, by a carpet; the area of this carpet is 280 square feet. Find the dimensions of the room.

39. A concert room seats 600 people, the chairs being disposed in rows right across the hall. Five chairs are taken out of each row to provide gangways, and it is found that, keeping the total number of chairs unaltered, the number of rows has to be increased by six. Find the original number of chairs in a row.

59. Simultaneous Equations with three or more Unknowns. If three unknowns x, y, z are connected by the two equations

$$x+y-z=8, \quad \text{.........................}(\alpha)$$
$$3x-2y+z=11, \quad \text{.........................}(\beta)$$

by addition it follows that

$$4x-y=19. \quad \text{.........................}(\gamma)$$

Thus x and y are connected by the single equation (γ). Giving to x the values 5, 6, 7, ... in succession, the corresponding values of y are 1, 5, 9, ... Giving to x and y in succession the values (6, 5), (7, 9), ..., and substituting these in either of the equations (α) or (β), the corresponding values of z are 3, 8, ...

Thus the integral solutions of (α) and (β) are (6, 5, 3), (7, 9, 8), ..., and the number of solutions is unlimited.

It will now be understood that the values of three unknowns cannot be determined by two equations.

The case where three unknowns x, y, z are connected by three equations will now be considered. In searching for a solution of three such equations, we may proceed by the method of substitution, or by the method of addition or subtraction; in either case we begin by assuming that a set of values can be found which satisfies the equations, and that x, y, z have these values.

60. Method by Substitution.

Ex. *Solve the equations*

$$4y+2z=3x, \quad \text{.........................}(\alpha)$$
$$2x+3y+4z=22, \quad \text{.........................}(\beta)$$
$$x+y-z=4. \quad \text{.........................}(\gamma)$$

From (α),
$$2z=3x-4y. \quad \text{.........................}(\delta)$$

From (γ),
$$2x+2y-2z=8. \quad \text{.........................}(\epsilon)$$

Substituting, in (β) and (ϵ), the value of $2z$ derived from (δ),

$$2x+3y+2(3x-4y)=22, \quad \text{.........................}(\zeta)$$
$$2x+2y-(3x-4y)=8. \quad \text{.........................}(\eta)$$

Simplifying (ζ) and (η),

$$8x-5y=22, \quad \text{.........................}(\theta)$$
$$6y-x=8. \quad \text{.........................}(\lambda)$$

Multiplying each side of (λ) by 8,

$$48y-8x=64. \quad \text{.........................}(\mu)$$

From (θ) and (μ), by addition, we have $43y=86$, $\therefore\ y=2$.

Substituting 2 for y in (λ), we find $x=4$.

Substituting 4 for x and 2 for y in (γ), we find $z=2$.

The above steps are reversible, hence the solution is (4, 2, 2)

61. Method by Addition or Subtraction.

Ex. *Solve the equations*

$$x+2y+3z=15, \quad \ldots\ldots(\alpha)$$
$$2x+3y-4z=9, \quad \ldots\ldots(\beta)$$
$$3x-y-5z=10. \quad \ldots\ldots(\gamma)$$

We choose an unknown, say y, and eliminate it between two pairs of the given equations, say between (α) and (γ) and between (β) and (γ).

Multiplying each side of (γ) by 2,

$$6x-2y-10z=20. \quad \ldots\ldots(\delta)$$

From (α) and (δ), by addition,

$$7x-7z=35,$$
$$\therefore\ x-z=5. \quad \ldots\ldots(\epsilon)$$

Multiplying each side of (γ) by 3,

$$9x-3y-15z=30. \quad \ldots\ldots(\zeta)$$

From (β) and (ζ), by addition,

$$11x-19z=39. \quad \ldots\ldots(\eta)$$

Multiplying each side of (ϵ) by 11,

$$11x-11z=55. \quad \ldots\ldots(\theta)$$

From (η) and (θ), by subtraction, $8z=16$, $\therefore\ z=2$.

Substituting 2 for z in (ϵ), we find $x=7$.

Substituting 7 for x and 2 for z in (γ), we have

$$3\,.\,7-y-5\,.\,2=10,$$
$$\therefore\ y=1.$$

The first step in this process consists in deriving equation (ϵ) from equations (α) and (γ) and in replacing (α) and (γ) by (ϵ) and (γ). This step is reversible, for equation (α) can be derived from equations (ϵ) and (γ). It can be shown (as in Art. 23) that all the other steps are reversible, hence (7, 1, 2) is the solution.

62. Exceptional Cases. Notice that in every case where an attempt is made to obtain the solution of a set of simultaneous equations, we begin by assuming that a solution exists.

Consider the equations $x+y=2$ and $x+y=3$. It is clear that these equations cannot be satisfied by the same values of x and y, for $x+y$ cannot stand for both 2 and 3 at the same time. Hence there is no solution, and the equations are said to be **inconsistent**.

Again, consider the equations $y-2x=3$ and $2(y-2x)=6$. One of these equations can be derived from the other by an algebraical process. The two equations are therefore equivalent to only one equation: the equations are therefore said to be **not independent**, and the number of solutions is unlimited.

Ex. 1. *Show that the equations*

$$x+y-z=8, \quad \ldots\ldots(\alpha)$$
$$3x-2y+z=11, \quad \ldots\ldots(\beta)$$
$$4x-y=19, \quad \ldots\ldots(\gamma)$$

are "not independent," and find two solutions.

From (α) and (β), by addition, $4x-y=19$; thus, equation (γ) can be derived from equations (α) and (β), and the given equations are not independent. Two solutions of these equations have been obtained in Art. 59, namely (6, 5, 3), (7, 9, 8).

Ex. 2. *Show that the equations*

$$x+y-z=8, \quad \ldots\ldots(\alpha)$$
$$3x-2y+z=11, \quad \ldots\ldots(\beta)$$
$$4x-y=13, \quad \ldots\ldots(\gamma)$$

are "inconsistent."

From (α) and (β), by addition, $4x-y=19$. Now $(4x-y)$ cannot denote 13 and 19 for the same values of x and y, hence the given equations are inconsistent.

EXERCISE XXVI.

Solve the following sets of simultaneous equations:

1. $x+y=23,$
$y+z=25,$
$z+x=24.$

2. $x+5y=46,$
$x+2y=22,$
$x+3z=33.$

3. $2x+7y=48,$
$5y-2x=24,$
$x+y+z=10.$

4. $x-(y+z)=4,$
$x+(y+z)=16,$
$y+5z=10.$

5. $x+y=3x+4y-23.$
$5x-3y=5,$
$x+2y+3z=35.$

6. $7x-5y=1,$
$4y-3z=1,$
$5z-8x=1.$

7. $x+y+z=9,$
$2x-y+z=8,$
$x-y=2z.$

8. $2x+3y=4z,$
$3x+4y=5z+4,$
$5x-3z=y-2.$

9. $y+z-x=19,$
$z+x-y=15,$
$x+y-z=7.$

10. $3x+2y-3z=4x-2z=5x-3y=10.$

11. $6(5x-6z)+3=3(2y+z-3x)+12-3y=5z.$

12. If $3x+7(y-z)=59$ and $5x-7(y-z)=5$, by what number does y exceed z?

13. Noticing the result in Ex. 12, explain why no solution exists for the equations

$$3x+7(y-z)=59, \quad 5x-7(y-z)=5, \quad y-z=10.$$

What are such equations called?

14. If $x+y+z=25$ and $6x-y+z=46$,

(i) prove that $5x-2y=21$.

(ii) By giving integral values to x, find solutions of the equation

$$5x-2y=21.$$

(iii) Find three integral solutions of the equations

$$x+y+z=25 \text{ and } 6x-y+z=46.$$

15. If $x+y-z=2$ and $5x-2y-2z=24$,

(i) prove that $3x-4y=20$.

(ii) Find three integral solutions of $3x-4y=20$.

(iii) Find three integral solutions of

$$x+y-z=2 \text{ and } 5x-2y-2z=24.$$

(iv) How many sets of values of x, y, z which are natural numbers satisfy the last two equations?

16. By choosing one of the unknowns and eliminating this letter between two pairs of equations, prove that

(i) $4x+3y-2z=4$, $5z-x-2y=10$, $4z+10x+5y=32$ are not independent equations.

(ii) The equations $x+2y-z=4$, $7x-5y+z=5$, $10x+y-2z=22$ are inconsistent.

(iii) The equations $x+y+z=23$, $4y+z-x=18$, $4x-y+2z=53$ are inconsistent.

17. A and B have £22 between them, and B and C together have £30, whilst C and A have £28. How much has each?

18. A, B, C, D are four stations on a line. The distance from A to C is 8 miles, and from B to D is 12 miles. Also the distance from C to D exceeds twice the distance from A to B by 1 mile. How far is D from C?

19. A and B are consecutive milestones on a straight road. X and Y are two points on the road such that XY exceeds AX by 100 yards and YB exceeds XY by 60 yards. Find the distance from Y to B in yards.

20. A number of three digits is such that if the order of the digits is reversed, the number is diminished by 99, also the sum of the digits is 14, and the middle digit is equal to the sum of the other two. Find the number.

21. When x has successively the values 1, 2 and 3, the corresponding values of the expression ax^2+bx+c are 18, 45 and 86. Find the values of a, b and c.

22. When x has successively the values 2, 3 and 4, the corresponding values of the expression ax^2-bx-c are 9, 30 and 61. Find the values of a, b and c.

23. When x and y have in succession the following values:

(i) $x=2, y=3$, (ii) $x=3, y=4$, (iii) $x=3, y=5$,

the corresponding values of the expression

$$xy-ax-by+c$$

are 3, 7 and 9. Find the values of a, b and c.

24. In a factory there are employed in one week 57 men, 14 women and 10 boys, the total wages for the week being £84. 5*s.*; the next week 60 men, 17 women and 11 boys are employed, the increase of wages amounting to £6. 5*s.*; if two women receive as much as a man and a boy, find the weekly earnings of a man, a woman and a boy respectively.

25. A dealer exchanges 10 sheep and 17 lambs for 5 pigs and £28. 9*s.* cash: in another exchange he gives 8 pigs for 2 sheep, 10 lambs and 12*s.* cash: in a third bargain he gives 3 lambs, 1 pig and £4. 15*s.* cash for 4 sheep: what is the value of each sheep, each lamb and each pig?

CHAPTER VIII.

LAWS OF DIVISION.

63. The Laws of Exact Division. If a and b stand for two natural numbers, and if a natural number x can be found such that $xb=a$, then a is said to be (exactly) divisible by b, and x is written in any of the forms $a \div b$, $\frac{a}{b}$ or a/b. (See Art. 15.)

Here we shall consider only the laws of exact division, and the letters *will be assumed to have such values that all the divisions indicated can be performed.* The operations indicated in $a.b.c$, $a.b \div c$, $a \div b \div c$ are to be conducted in order from left to right (unless it has been proved that the order does not affect the result), and a bracket is to have its usual meaning: thus $(a \div b)c$ means the same number as $a \div b.c$, and $a \div (bc)$ means that a is to be divided by the product bc.

From the definition of division, it follows that the equations $a=bc$, $a \div b=c$, $a \div c=b$ all mean the same thing; thus the equation $a=bc$ can be written in either of the forms

$$\frac{a}{b}=c \text{ or } \frac{a}{c}=b.$$

In particular, from the equation $a=a.1$, it follows that $\frac{a}{a}=1$ and $\frac{a}{1}=a$.

The Fundamental Laws of Multiplication and Division are the **Commutative**, the **Associative** and the **Distributive Laws**, and on these laws all such operations depend. These laws are considered in Arts. 65, 66, 69.

64. Multiplication and Division are Inverse Operations. By the definition of division,

$$(ab \div b)b = ab = (a \div b.b)b;$$

$$\therefore \mathbf{ab \div b = a = a \div b.b.}$$

Thus, to multiply a by b and divide the result by b, or to divide a by b and multiply the result by b, is to leave the number a unaltered. Hence, **multiplication and division are inverse operations.**

65. The Commutative Law. This law asserts that:

In performing a series of operations, which may be either multiplications or divisions, the result is not affected by the order in which the operations are conducted. Thus

$$\text{(i)}\quad \mathbf{a \div b . c = a . c \div b} \quad \text{or} \quad \mathbf{\frac{a}{b} . c = \frac{ac}{b}},$$

$$\text{(ii)}\quad \mathbf{a \div b \div c = a \div c \div b} \quad \text{or} \quad \mathbf{\frac{\frac{a}{b}}{c} = \frac{\frac{a}{c}}{b}}.$$

Proof of (i). Let $a \div b = x$; then, by hypothesis, x is a natural number, and

$$a = xb. \qquad (\textit{Def. of Division})$$

Multiplying each side by c,

$$ac = xbc = (xc)b.$$

Dividing each side by b,

$$ac \div b = xc = a \div b . c.$$

Proof of (ii). It will first be shown that $a \div b \div c = a \div (bc)$.

Let $a \div (bc) = x$; then, assuming that x is a natural number, we have

$$a = x(bc) = (xc)b.$$

Dividing each side by b,

$$a \div b = xc.$$

Dividing each side by c,

$$a \div b \div c = x = a \div (bc).$$

It can be proved in a similar manner that

$$a \div c \div b = a \div (cb),$$

and since $bc = cb$, it follows that $a \div b \div c = a \div c \div b$.

66. The Associative Law. The Associative Law for addition and subtraction is contained in the identities:

$$\mathbf{a + (b + c) = a + b + c}, \qquad \mathbf{a - (b + c) = a - b - c},$$

$$\mathbf{a + (b - c) = a + b - c}, \qquad \mathbf{a - (b - c) = a - b + c}.$$

If in these identities we replace the signs for addition and subtraction by the signs for multiplication and division, respec-

tively, we shall obtain a statement of the **Associative Law for Multiplication and Division.** This law is then contained in the formulae :

(i) $\mathbf{a(bc)=abc}$,

(ii) $\mathbf{a\div(bc)=a\div b\div c}$ or $\mathbf{\dfrac{a}{bc}=\dfrac{\frac{a}{b}}{c}}$,

(iii) $\mathbf{a.(b\div c)=a.b\div c}$ or $\mathbf{a.\dfrac{b}{c}=\dfrac{ab}{c}}$,

(iv) $\mathbf{a\div(b\div c)=a\div b.c}$ or $\mathbf{\dfrac{a}{\frac{b}{c}}=\dfrac{a}{b}.c}$.

The *Proof* of (i) is given in Art. 26 and that of (ii) in Art. 65 (ii).

Proof of (iii). Let $b\div c=x$; then, by hypothesis, x is a natural number, and

$$b=xc. \qquad \textit{(Def. of Division)}$$

Multiplying each side by a,

$$ba=xca;$$

$$\therefore\ ab=axc. \qquad \textit{(Commutative Law for Multiplication)}$$

Dividing each side by c,

$$ab\div c=ax=a.(b\div c).$$

Proof of (iv). Let $b\div c=x$; then, assuming that x is a natural number,

$$b=xc. \qquad \textit{(Def. of Division)}$$

Hence it follows that $a\div b=a\div(xc)$; *(Rule for Equalities)*

$$\therefore\ a\div b=a\div x\div c. \qquad \textit{(Identity (ii) above)}$$

Multiplying each side by c,

$$a\div b.c=a\div x=a\div(b\div c).$$

It may now be shown that **if all the divisions indicated can be performed,** *the brackets can be removed from such expressions as $a.(b.c\div d\div e)$ and $a\div(b.c\div d\div e)$ by a process similar to that of removing the brackets from $a+(b+c-d-e)$ and $a-(b+c-d-e)$, the sign $\times$ replacing $+$, and $\div$ replacing $-$; thus*

$$a.(b.c\div d\div e)=a.b.c\div d\div e,$$
$$a\div(b.c\div d\div e)=a\div b\div c.d.e.$$

Also, the Commutative Law, as stated in Art. 65, can now be established for any number of multiplications and divisions.

67. Important Theorems on Exact Division.

Theorem 1. $$\frac{a}{b}=\frac{ax}{bx}.$$

Proof.
$$\begin{aligned}(ax)\div(bx)&=a\,.\,x\div b\div x && \textit{(Associative Law)}\\ &=a\,.\,x\div x\div b && \textit{(Commutative Law)}\\ &=a\div b\,; && \textit{(Art. 64)}\end{aligned}$$
$$\therefore\quad \frac{a}{b}=\frac{ax}{bx}.$$

Such an expresion as $\frac{ax}{bx}$ may therefore be simplified by *removing any factor which is common to the dividend and divisor.*

Theorem 2. $$\frac{a}{b}\cdot\frac{x}{y}=\frac{ax}{by}.$$

Proof.
$$\begin{aligned}(a\div b)\,.\,(x\div y)&=a\div b\,.\,x\div y && \textit{(Associative Law)}\\ &=a\,.\,x\div b\div y && \textit{(Commutative Law)}\\ &=(ax)\div(by)\,; && \textit{(Associative Law)}\end{aligned}$$
$$\therefore\quad \frac{a}{b}\cdot\frac{x}{y}=\frac{ax}{by}.$$

Theorem 3. $$\frac{a}{b}\div\frac{x}{y}=\frac{ay}{bx}.$$

Proof.
$$\begin{aligned}(a\div b)\div(x\div y)&=a\div b\div x\,.\,y && \textit{(Associative Law)}\\ &=a\div(bx)\,.\,y && \textit{(Associative Law)}\\ &=(ay)\div(bx)\,; && \textit{(Commutative and Associative Laws)}\end{aligned}$$
$$\therefore\quad \frac{a}{b}\div\frac{x}{y}=\frac{ay}{bx}.$$

68. Index Laws. The following examples are instances of Fundamental Index Laws:

Ex. 1. *Prove that* $x^5\div x^2=x^{5-2}$.
$$\begin{aligned}x^5\div x^2&=(xxxxx)\div(xx) && \textit{(Def.)}\\ &=(xxx)(xx)\div(xx) && \textit{(Associative Law)}\\ &=xxx\\ &=x^{5-2}.\end{aligned}$$

Ex. 2. *Prove that* $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$.

$$\begin{aligned}(x \div y)^2 &= (x \div y)(x \div y) && (Def)\\ &= x \div y \,.\, x \div y && (Associative\ Law)\\ &= x \,.\, x \div y \div y && (Commutative\ Law)\\ &= (xx) \div (yy) && (Associative\ Law)\\ &= x^2 \div y^2.\end{aligned}$$

The argument in the above examples does not depend on the particular numbers chosen as indices so long as the operations indicated are *possible*. We are therefore able to state the following **Fundamental Index Laws** :

If m is greater than n, then

$$\text{(i)}\ \frac{\mathbf{x^m}}{\mathbf{x^n}} = \mathbf{x^{m-n}}, \qquad \text{(ii)}\ \left(\frac{\mathbf{x}}{\mathbf{y}}\right)^{\mathbf{n}} = \frac{\mathbf{x^n}}{\mathbf{y^n}}.$$

Ex. 1. *Simplify* $\frac{9a^3b}{6a^2b^4}$.

The H.C.F. of the dividend and divisor is $3a^2b$, and

$$\frac{9a^3b}{6a^2b^4} = \frac{(3a)(3a^2b)}{(2b^3)(3a^2b)} = \frac{3a}{2b^3}. \qquad (Art.\ 67,\ Theorem\ 1)$$

Ex. 2. *Simplify* $\frac{15x^2y^3}{12a^2b} \div \frac{5x^3y}{6a^5}$.

$$\begin{aligned}\text{The given expression} &= \frac{(15x^2y^3)(6a^5)}{(12a^2b)(5x^3y)} && (Art.\ 67,\ Theorem\ 3)\\ &= \frac{15 \,.\, 6 \,.\, a^5x^2y^3}{5 \,.\, 12 \,.\, a^2bx^3y}\\ &= \frac{3a^3y^2}{2bx}. && (As\ in\ \text{Ex. } 1)\end{aligned}$$

Ex. 3. *Simplify*

(i) $\frac{(a+b)^2}{a^2-b^2}$. (ii) $\frac{6a^2+12ab}{a^2-4b^2}$. (iii) $\frac{(a^3b-ab^3)^2}{(2a+2b)^2}$.

$$\text{(i)}\ \frac{(a+b)^2}{a^2-b^2} = \frac{(a+b)^2}{(a-b)(a+b)} = \frac{a+b}{a-b}.$$

$$\text{(ii)}\ \frac{6a^2+12ab}{a^2-4b^2} = \frac{6a(a+2b)}{(a-2b)(a+2b)} = \frac{6a}{a-2b}.$$

$$\text{(iii)}\quad \frac{(a^3b-ab^3)^2}{(2a+2b)^2}=\frac{\{ab(a^2-b^2)\}^2}{\{2(a+b)\}^2}=\frac{a^2b^2(a^2-b^2)^2}{4(a+b)^2}$$
$$=\frac{a^2b^2(a-b)^2(a+b)^2}{4(a+b)^2}$$
$$=\frac{a^2b^2(a-b)^2}{4}.$$

NOTE. The expressions $\frac{a+b}{a-b}$, $\frac{6a}{a-2b}$, $\frac{a^2b^2(a-b)^2}{4}$ cannot be further simplified unless the values of a and b are known: here, these values are supposed to be such that the divisions can be performed.

EXERCISE XXVII.

NOTE. *Letters, of which the values are not given, are supposed to have such values that the operations indicated are possible.*

Simplify

1. $\frac{2ab}{2b}$. 2. $\frac{2x^4y^5}{4x^3y^2}$. 3. $\frac{6(x^2y^3)^2}{2x^4y^4}$. 4. $\frac{(3ab^2)^3}{9a^2b^5}$.

5. $\frac{2(xy^2)^3}{3(x^2y)^3}$. 6. $\frac{25a^2b^3c^4}{15a^4b^5c}$. 7. $\frac{14a^3b^5c^2}{21ab^2c^5}$. 8. $\frac{10a^2(b+c)^3}{5a^3(b+c)^2}$.

9. $\frac{x^2y^3(x+y)^2(x-y)^2}{xy(x+y)(x-y)^3}$. 10. $\frac{x^3(2x+y)}{2(x+y)x^2}$. 11. $\frac{2ab}{3c}\times\frac{4cd}{8b}$.

12. $\frac{15x^2y}{6z^2}\times\frac{8yz^2}{10xy^2}$. 13. $\frac{4xy^2z^3}{2x^2y^2z^2}\div\frac{3xy}{6y^2z^2}$. 14. $\frac{3ab}{a^2+ab}$. 15. $\frac{4ab^2}{b^3-ab}$.

16. $\frac{x^3y^2+x^2y^3+2xy}{xy}$. 17. $\frac{(x^2-x)^2}{x^2}$. 18. $\frac{ab^2-a^2b}{b-a}$. 19. $\frac{a^2b+ab^2}{a+b}$.

20. $\frac{x^3(x+2)-x-2}{x+2}$. 21. $\frac{2x^2(x-1)-x+1}{x-1}$. 22. $\frac{(x-2)^2-2x+4}{x-2}$.

23. $\frac{xb-xc-yb+yc}{b-c}$. 24. $\frac{x(b-c)+y(b-c)+z(b-c)}{b-c}$.

25. Remove the brackets from

(i) $bd\left(\frac{a}{b}+\frac{c}{d}\right)$. (ii) $12\left(\frac{x}{2}+\frac{y}{3}-\frac{z}{4}\right)$. (iii) $abc\left(\frac{2x}{a}+\frac{3y}{b}-\frac{4z}{c}\right)$.

(iv) $x^2y^2\left(\frac{5x^2}{y^2}+\frac{6y^2}{x^2}\right)$. (v) $10\left(\frac{6x}{5}-\frac{5y}{6}\right)$.

26. Simplify

(i) $\frac{a^2}{b^2} \times \frac{bc}{ad} \div \frac{ab}{cd} \div \frac{c^2}{d^2}$. (ii) $\left(\frac{a^2}{b^2} \times \frac{bc}{ad}\right) \div \left(\frac{ab}{cd} \div \frac{c^2}{d^2}\right)$.

(iii) $\frac{x^2}{y^2} \div \frac{yz}{xw} \div \left(\frac{xy}{zw} \div \frac{z^2}{w^2}\right)$. (iv) $\frac{x^2}{y^2} \div \left(\frac{yz}{xw} \div \frac{xy}{zw}\right) \times \frac{z^2}{w^2}$.

(v) $\left(\frac{l^2}{m^2} \div \frac{mn}{lp}\right) \times \left(\frac{lm}{np} \div \frac{n^2}{p^2}\right)$. (vi) $\frac{l^2}{m^2} \div \left(\frac{mn}{lp} \times \frac{lm}{np} \times \frac{n^2}{p^2}\right)$.

27. By means of the identity

$$a^2 - b^2 = (a+b)(a-b),$$

simplify the following:

(i) $\frac{x^2-1}{x+1}$. (ii) $\frac{3x^2-3}{2x-2}$. (iii) $\frac{4x^2-1}{4x-2}$. (iv) $\frac{(1+x)^2-4}{x+3}$.

(v) $\frac{(x+1)^2-(x-1)^2}{2x}$. (vi) $\frac{(y-z)^2-x^2}{y-(z+x)}$. (vii) $\frac{a^3b-ab^3}{a-b}$.

(viii) $\frac{9a^4b^2-4a^2b^4}{3a^3b^2+2a^2b^3}$. (ix) $\frac{(a^2b+ab^2)^2}{(a^2-b^2)^2}$. (x) $x+y-\frac{9x^2-4y^2}{3x+2y}$.

28. By means of the identities

$$(a+b)^2 = a^2+2ab+b^2$$

$$\text{and } (a-b)^2 = a^2-2ab+b^2,$$

simplify the following expressions:

(i) $\frac{6(a+b)^3}{2a^2+4ab+2b^2}$. (ii) $\frac{(2a-b)^3}{4a^2-4ab+b^2}$.

(iii) $\frac{9(a^2-b^2)^2}{3a^2-6ab+3b^2}$. (iv) $\frac{(x^2+5yz)(2x^2+10yz)}{x^4+10x^2yz+25y^2z^2}$.

(v) $\frac{a^2+2ab+b^2}{a+b} - \frac{a^2-2ab+b^2}{a-b}$.

29. Of what expressions are the following the squares? Verify the results by substituting 1 for x:

(i) $1+\frac{12}{x}+\frac{36}{x^2}$. (ii) $25x^2+\frac{16}{x^2}-40$. (iii) $(x+1)^2+\frac{100}{(x+1)^2}-20$.

30. By means of the identities

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$\text{and}\quad a^3-b^3=(a-b)(a^2+ab+b^2),$$

simplify the following expressions :

(i) $\dfrac{a^3+b^3}{a^2-ab+b^2}-\dfrac{a^3-b^3}{a^2+ab+b^2}$. (ii) $\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}$.

(iii) $\dfrac{64x^3+27y^3}{4x+3y}$. (iv) $\dfrac{8x^3-1}{4x^2+2x+1}$. (v) $\dfrac{a^4b+ab^4}{a^4-a^3b+a^2b^2}$.

31. Simplify (i) $\dfrac{10x^2-9y(x+y)}{5x^2-y(7x+6y)}$; (ii) $\dfrac{2x^3-3x^2+4x-6}{x^3-2x^2+2x-4}$.

69. The Distributive Law. The complete statement of this Law is as follows :

(i) $\mathbf{(a+b)c=(ac)+(bc)}$. (ii) $\mathbf{(a+b)\div c=(a\div c)+(b\div c)}$.

(iii) $\mathbf{(a-b)c=(ac)-(bc)}$. (iv) $\mathbf{(a-b)\div c=(a\div c)-(b\div c)}$.

(v) $\mathbf{c(a+b)=(ca)+(cb)}$. (vi) $\mathbf{c(a-b)=(ca)-(cb)}$.

Identities (ii) and (iv) may also be stated thus :

(ii) $\mathbf{\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}}$. (iv) $\mathbf{\dfrac{a-b}{c}=\dfrac{a}{c}-\dfrac{b}{c}}$.

Identities (i), (iii), (v), (vi) have been proved in Arts. 12, 33.

Proof of (ii). By hypothesis $a\div c$ and $b\div c$ are natural numbers,

$$\therefore\ (a\div c+b\div c)c=a\div c\,.\,c+b\div c\,.\,c$$
$$=a+b. \qquad (\textit{Def. of Division})$$

Dividing each side by c,

$$a\div c+b\div c=(a+b)\div c.$$

The proof of (iv) is similar to that of (ii).

It is important to observe that if a, b, c stand for natural numbers,

$\mathbf{c\div(a+b)}$ *is* not *equal to* $\mathbf{(c\div a)+(c\div b)}$,

i.e. $\mathbf{\dfrac{c}{a+b}}$ *is* not *equal to* $\mathbf{\dfrac{c}{a}+\dfrac{c}{b}}$.

In order to prove this, it is sufficient to show that the expressions are not equal for *any particular* set of values of c, a, b.

Thus, if $c=24$, $a=2$, $b=6$, we have

$$c \div (a+b) = 24 \div (2+6) = 24 \div 8 = 3,$$
$$(c \div a) + (c \div b) = (24 \div 2) + (24 \div 6) = 12 + 4 = 16;$$
$$\therefore\ c \div (a+b) \text{ is not equal to } (c \div a) + (c \div b),$$

that is to say, $\frac{c}{a+b}$ is not equal to $\frac{c}{a} + \frac{c}{b}$.

In the same way it can be shown that

$$\frac{\mathbf{c}}{\mathbf{a-b}}\ \textit{is}\ \textbf{not}\ \textit{equal to}\ \frac{\mathbf{c}}{\mathbf{a}} - \frac{\mathbf{c}}{\mathbf{b}}.$$

70. Important Theorems.

Theorem 1. $$\frac{a}{b} + \frac{x}{y} = \frac{ay+bx}{by}.$$

Proof. $$\frac{a}{b} = \frac{ay}{by} \text{ and } \frac{x}{y} = \frac{xb}{yb} = \frac{bx}{by}; \quad (\textit{Art. } 67, \textit{ Th. } 1)$$

$$\therefore\ \frac{a}{b} + \frac{x}{y} = \frac{ay}{by} + \frac{bx}{by} \quad (\textit{Rule for Equalities})$$

$$= \frac{ay+bx}{by}. \quad (\textit{Distributive Law})$$

Theorem 2. $$\frac{a}{b} - \frac{x}{y} = \frac{ay-bx}{by}.$$

The proof is similar to that of Theorem 1.

NOTE. In theorems (1), (2) we have expressed $\frac{a}{b} + \frac{x}{y}$ and $\frac{a}{b} - \frac{x}{y}$ each **as a single term.**

Ex. 1. *Express each of the following as a single term:*

(i) $\frac{a}{x} + \frac{b}{x} - \frac{2}{x}$. (ii) $5 - \frac{12}{x}$. (iii) $\frac{x}{a} - \frac{y}{b} + \frac{z}{c}$. (iv) $a + b + \frac{b^2}{a-b}$.

(i) $$\frac{a}{x} + \frac{b}{x} - \frac{2}{x} = \frac{a+b-2}{x}. \quad (\textit{Distributive Law})$$

(ii) $$5 - \frac{12}{x} = \frac{5x}{x} - \frac{12}{x} = \frac{5x-12}{x}.$$

(iii) $$\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=\frac{xbc}{abc}-\frac{yca}{abc}+\frac{zab}{abc}=\frac{xbc-yca+zab}{abc}.$$

(iv) $$a+b+\frac{b^2}{a-b}=\frac{(a+b)(a-b)}{a-b}+\frac{b^2}{a-b}$$
$$=\frac{a^2-b^2}{a-b}+\frac{b^2}{a-b}=\frac{a^2-b^2+b^2}{a-b}=\frac{a^2}{a-b}.$$

Ex. 2. *Simplify* $\dfrac{3x^2+4xy}{4x}-\dfrac{xy+y^2}{3y}$.

The given expression $$=\frac{x(3x+4y)}{4x}-\frac{y(x+y)}{3y}$$
$$=\frac{3x+4y}{4}-\frac{x+y}{3}$$
$$=\frac{3(3x+4y)}{12}-\frac{4(x+y)}{12}\,{}^{*}$$
$$=\frac{(9x+12y)-(4x+4y)}{12}$$
$$=\frac{5x+8y}{12}.$$

* In this step the number 12 is chosen as being the L.C.M. of 4 and 3.

Ex. 3. *Simplify* (i) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$. (ii) $\dfrac{y^3\left(\frac{12x^2}{y^2}-3\right)}{\frac{2x}{y}-1}$.

(i) $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}=\frac{\frac{ad+bc}{bd}}{\frac{ad-bc}{bd}}=\frac{\left(\frac{ad+bc}{bd}\right)bd}{\left(\frac{ad-bc}{bd}\right)bd}=\frac{ad+bc}{ad-bc},$$

or thus $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}=\frac{\left(\frac{a}{b}+\frac{c}{d}\right)bd}{\left(\frac{a}{b}-\frac{c}{d}\right)bd}=\frac{ad+bc}{ad-bc}.$$

(ii) $$\frac{y^3\left(\frac{12x^2}{y^2}-3\right)}{\frac{2x}{y}-1}=\frac{3y^3\left(\frac{4x^2}{y^2}-1\right)}{\frac{2x}{y}-1}=\frac{3y^3\left(\frac{2x}{y}+1\right)\left(\frac{2x}{y}-1\right)}{\frac{2x}{y}-1}$$

$$=3y^3\left(\frac{2x}{y}+1\right)$$

$$=3y^2(2x+y).$$

EXERCISE XXVIII.

Express each of the following as a single term :

1. $1+\frac{6}{x}.$ **2.** $a-\frac{b}{c}.$ **3.** $2x+\frac{6}{y}.$

4. $2x^2+\frac{4}{x}.$ **5.** $3x^3+\frac{12}{x^2}.$ **6.** $a-\frac{b}{c}+\frac{x}{y}.$

7. $x-\frac{3}{x}+\frac{9}{x^2}.$ **8.** $\frac{4}{x}+2+x.$ **9.** $\frac{x}{2a}+\frac{y}{3a}-\frac{z}{4a}.$

10. $\frac{2x}{3a}+\frac{3x}{4a}+\frac{x}{a}.$ **11.** $a+1+\frac{6}{a-1}.$ **12.** $a+3b+\frac{9b^2}{a-3b}.$

Simplify the following :

13. $x-\frac{x-1}{2}+\frac{x-2}{3}.$ **14.** $\frac{a+2b}{3}-\frac{a-3b}{4}.$

15. $\frac{2a^2+3ab}{4a}-\frac{3b^2-4ab}{3b}.$ **16.** $3\,.\,\frac{2x+y}{5}-2\,.\,\frac{2x-y}{3}.$

17. $\frac{6(3x+2y)}{7}-\frac{5(3x-y)}{2}.$ **18.** $\frac{x+2y}{2}+\frac{3x+y}{3}+\frac{2x+3y}{4}.$

19. $\frac{x-1}{2}+\frac{3x-1}{4}-\frac{5x-1}{6}.$ **20.** $\frac{a-3c}{4}-\frac{3(c-2b)}{8}-\frac{3a-5b}{12}.$

21. $\frac{a-2}{2}-\frac{a-3}{3}+\frac{a-4}{4}-\frac{a-6}{6}.$ **22.** $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}.$

23. $\frac{4a}{3x-3}+\frac{3a}{4x-4}.$ **24.** $\frac{7a}{4x+8y}-\frac{3a}{2x+4y}.$

25. $\frac{a}{x+1}+\frac{3a}{2x+2}-\frac{5a}{4x+4}.$ **26.** $\frac{x^2}{x-3y}+\frac{3x^2y}{(x-3y)^2}.$

27. $a+b-\frac{b(2a+b)}{a+b}.$ **28.** $a-b+\frac{b(2a-b)}{a-b}.$

29. $2a+b-\frac{4a(a+b)}{2a+b}.$ **30.** $3a-2b+\frac{3a(4b-3a)}{3a-2b}.$

31. $\dfrac{6}{a+1}+\dfrac{9}{(a+1)^2}$.

32. $\dfrac{12}{a-1}-\dfrac{4a}{a^3-2a^2+a}$.

33. $x+y+\dfrac{y^3}{x^2-xy+y^2}$.

34. $x^2+xy+y^2+\dfrac{x^3+y^3}{x-y}$.

35. $2a+b-\dfrac{8a^3}{4a^2-2ab+b^2}$.

36. $a^2+2ab+b^2-\dfrac{a(a^2+3ab+4b^2)}{a+b}$.

37. $a^2-2ab+b^2+\dfrac{b(3a^2-3ab+b^2)}{a-b}$.

38. $\dfrac{(a+b)^4-(a-b)^4}{4ab}-4ab$.

39. $\dfrac{a+\dfrac{b}{4}}{\dfrac{b}{8}}$.

40. $\dfrac{x(x+y)^2}{1+\dfrac{x}{y}}$.

41. $\dfrac{(x-y)^2}{\dfrac{x}{y}-1}$.

42. $\dfrac{\dfrac{1}{y^2}(x^3+y^3)}{\dfrac{x^2}{y^2}-\dfrac{x}{y}+1}$.

43. $\dfrac{81x^2-\dfrac{16}{x^2}}{9-\dfrac{4}{x^2}}$.

44. $\dfrac{yz+\dfrac{a}{x}}{x+\dfrac{a}{yz}}$.

45. $\left(1+\dfrac{3x}{y}+\dfrac{3x^2}{y^2}+\dfrac{x^3}{y^3}\right)\div\dfrac{(x+y)^2}{y^3}$.

46. Express by means of algebraical symbols the following directions:

"Subtract the product of $2a+3b$ and $3a+4b$ from the product of $3a+2b$ and $4a+3b$, and divide the difference by three times the sum of a and b."

Find the result in its simplest form.

71. Rules for Equalities and Inequalities.

If a is divisible by b and x is divisible by y, then

$$\frac{a}{b} > = \text{ or } < \frac{x}{y}$$

according as $\qquad$ **ay > = or < xb.**

Proof. (i) Let $ay > xb$; then since the character of this inequality is unaltered if each side is divided by by, which is the same as yb, (*Rule* 6 *for Inequalities, Art.* 22)

$$\therefore \frac{ay}{by} > \frac{xb}{yb};$$

$$\therefore \frac{a}{b} > \frac{x}{y}.$$

(ii) Let $ay = xb$; dividing each side of this equality by by,

$$\frac{ay}{by} = \frac{xb}{yb}; \qquad \therefore \ \frac{a}{b} = \frac{x}{y}.$$

(iii) Let $ay < xb$, then as in (i) it can be shown that $\frac{a}{b} < \frac{x}{y}$.

Conversely, ay > = or < xb according as $\mathbf{\frac{a}{b}}$ > = or < $\mathbf{\frac{x}{y}}$.

This follows from the preceding, or we may proceed as follows:

Proof. (i) Let $\frac{a}{b} > \frac{x}{y}$; then since the character of this inequality is unaltered if each side is multiplied by by (which is the same as yb),

$$\frac{a}{b}(by) > \frac{x}{y}(yb); \qquad \therefore \ ay > xb.$$

The other cases may be proved in a similar manner.

Ex. 1. *Find x in terms of a from the equation* $\frac{6}{x} = \frac{a}{2}$.

Multiplying each side of the given equation by $2x$,

$$\frac{6}{x}(2x) = \frac{a}{2}(2x); \qquad \therefore \ 12 = ax.$$

Dividing each side by a, $\qquad \frac{12}{a} = x; \quad \therefore \ x = \frac{12}{a}.$

This process is reversible; hence $\frac{12}{a}$ is a solution, and is the only solution.

Ex. 2. *Solve the equation*

$$x + \frac{3x+1}{5} - \frac{2x-1}{3} = 12 - \frac{x+16}{6}.$$

Multiply each side by 30, the L.C.M. of the divisors 5, 3, 6.

$$30x + \tfrac{30}{5}(3x+1) - \tfrac{30}{3}(2x-1) = 360 - \tfrac{30}{6}(x+16);^{*}$$

$$\therefore \ 30x + 6(3x+1) - 10(2x-1) = 360 - 5(x+16),$$

and the solution can be completed as in Art. 47.

---

* This step may be omitted in practice.

72. Fractions. When a is not exactly divisible by b, $\frac{a}{b}$ belongs to a new class of numbers called fractions, whose properties will be considered later. Here, it is sufficient to remark that in Arithmetic, $\frac{1}{5}$ *of* 10 units means the same as $(10 \div 5)$ units and $\frac{3}{5}$ *of* 10 units means the same as $(10 \div 5 \times 3)$ units. The expression $\frac{1}{5}(x+2)$ will be taken to mean the same as $(x+2) \div 5$ and $\frac{3}{5}(x+2)$ will mean the same as $(x+2) \div 5 \times 3$. In all such cases, it will be assumed for the present, that the letters have such values that the divisions can be performed.

Ex. 1. *Solve the simultaneous equations*

$$\frac{1}{7}(2x-3y)+\frac{1}{2}(3x-5y)=6, \quad \dots\dots\dots (\alpha)$$

$$\frac{3}{4}(x+3y)-\frac{2}{3}(x+y)=2. \quad \dots\dots\dots (\beta)$$

Multiplying each side of (α) by 14,

$$2(2x-3y)+7(3x-5y)=84.$$

Simplifying this equation, we have

$$25x-41y=84. \quad \dots\dots\dots (\gamma)$$

Multiplying each side of (β) by 12,

$$9(x+3y)-8(x+y)=24.$$

Simplifying this equation, we have

$$x+19y=24. \quad \dots\dots\dots (\delta)$$

From equations (γ), (δ), we find that $x=5$ and $y=1$; the solution is therefore (5, 1).

Ex. 2. *Find x and y in terms of a and b from the equations*

$$\frac{x}{a}+\frac{y}{b}=a+b, \quad \dots\dots\dots (\alpha)$$

$$x-y=a^2-b^2. \quad \dots\dots\dots (\beta)$$

Multiplying each side of (α) by ab,

$$bx+ay=ab(a+b). \quad \dots\dots\dots (\gamma)$$

Multiplying each side of (β) by a,

$$ax-ay=a(a^2-b^2). \quad \dots\dots\dots (\delta)$$

From (γ) and (δ) by addition,

$$ax + bx = ab(a + b) + a(a^2 - b^2);$$

$$\therefore \ x(a + b) = (a + b)\{ab + a(a - b)\};$$

$$\therefore \ x(a + b) = (a + b)a^2;$$

$$\therefore \ x = a^2.$$

Substituting a^2 for x in (β) we see that

$$y = b^2.$$

EXERCISE XXIX.

Find x in terms of the other letters from the equations :

1. $\dfrac{x}{a} = \dfrac{b}{c}$. **2.** $\dfrac{3}{x} = \dfrac{8}{c}$. **3.** $\dfrac{5}{9} = \dfrac{x}{a}$. **4.** $\dfrac{6}{7} = \dfrac{a}{x}$.

5. $\dfrac{x}{a} + 1 = \dfrac{b}{c}$. **6.** $\dfrac{x}{a} + \dfrac{b}{c} = 1$. **7.** $\dfrac{12}{x} = \dfrac{6}{a} - 1$. **8.** $\dfrac{x - a}{a} = \dfrac{b - c}{c}$.

Solve the following equations :

9. $\dfrac{1}{2}x - \dfrac{1}{3}x + 7 = \dfrac{5x}{6} - 5$. **10.** $\dfrac{2x + 1}{3} - \dfrac{x + 6}{5} = x - 3$.

11. $\dfrac{x - 11}{12} + \dfrac{x - 12}{11} = 2$. **12.** $\dfrac{x + 2}{3} - \dfrac{x + 1}{5} = \dfrac{x - 3}{4} - 1$.

13. $\dfrac{1}{3}(x - 1) + \dfrac{3}{4}(x + 1) = \dfrac{1}{6}(5x + 13)$.

14. $x - \dfrac{x - 2}{3} = \dfrac{x + 23}{4} - \dfrac{10 + x}{5}$. **15.** $\dfrac{x + 2}{3} + \dfrac{x + 3}{4} = \dfrac{x + 8}{9} + \dfrac{x + 9}{10}$.

16. $\dfrac{1}{2}(x + 1) + \dfrac{1}{3}(x + 2) + \dfrac{1}{4}(x + 3) = 16$.

17. $\dfrac{1}{6}x + \dfrac{1}{5}(2x + 1) - \dfrac{1}{4}(x - 4) = 2(x - 9) - \dfrac{1}{12}x$.

18. $\dfrac{1}{4}(5x - 11) - \dfrac{1}{10}(x - 1) = \dfrac{1}{12}(11x - 1)$.

19. $x + \dfrac{3x + 1}{5} - \dfrac{2x - 1}{3} = 12 - \dfrac{x + 16}{6}$.

20. $\dfrac{3}{5}(2x - 1) + \dfrac{1}{4}(3x - 7) = \dfrac{5}{6}(x + 5) + \dfrac{x + 3}{2}$.

21. $\dfrac{3}{4}(3x + 1) - \dfrac{2}{5}(4x - 5) = x + 1$.

22. $\frac{3(2x-9)}{11}-\frac{2(x-11)}{7}=\frac{4x-47}{9}$.

23. $\frac{2}{5}(7x-3)-\frac{3}{7}(5x+1)=\frac{2}{3}(x+2)-3$.

24. $\frac{1}{2}(2x-1)(x+2)-\frac{1}{3}(3x-2)(x+1)=2$.

25. $x^2+\frac{5}{7}(7x-8)(x+3)-\frac{3}{5}(2x-3)(5x+2)=10(x+1)$.

26. $\frac{5}{8}(8x-9)(x+3)-\frac{2}{3}(3x+1)(2x-7)=(x+5)^2+23$.

27. $\frac{18}{25}(x+1)(5x-2)(5x-3)-\frac{1}{2}(3x-2)(2x-1)(6x+7)=5x-7$.

28. $(x-1)^3-\frac{1}{2}(x-2)^3-\frac{1}{8}(x-8)(2x+7)(2x+9)=25x+4$.

29. $3x-\frac{y}{2}=5,$
$\frac{x}{3}+\frac{y}{4}=3.$

30. $\frac{x}{2}+\frac{y}{3}=20,$
$\frac{x}{4}+\frac{y}{5}=11.$

31. $4x-5y-1=\frac{1}{2}(7x-8y-3)=10$.

32. $3x-\frac{1}{2}(y-2x)=1$; $2y-\frac{3}{2}(10x-y)=19$.

33. $\frac{7x+3}{5}=\frac{5y-7}{4}=\frac{2x+3y-5}{3}$.

34. $\frac{x-1}{8}+\frac{y-2}{5}=2,$
$2x+\frac{2y-5}{3}=21.$

35. $x-\frac{x-y}{3}=y+4,$
$2y+\frac{2x-3y}{4}=20-x.$

36. $\frac{x-11}{7}-\frac{y-6}{5}=8,$
$\frac{x-12}{3}+\frac{y-4}{2}=29.$

37. $\frac{3x-y}{8}+2=\frac{4x-y}{7},$
$\frac{3y-x}{4}+1=\frac{4y-x}{5}.$

38. $(x+2)(y-3)-(x+4)(y-5)=10,$
$$\frac{7y-6x}{3}-\frac{5x-3y}{4}=5.$$

39. $x-\frac{1}{3}(y-2x-1)=\frac{x-1}{3}+y-\frac{30x}{7}=2$.

Find the values of x in terms of the other letters from the equations.

40. $ax+b=3ax+c.$

41. $\frac{x-a}{b}=\frac{x-b}{a}.$

42. $\frac{a+x}{b}+\frac{b-x}{a}=2.$

43. $x+\frac{9b^2}{a}=\frac{3bx}{a}+a.$

44. $x-\frac{x-a}{b+1}=a-\frac{x}{b+1}.$

45. $\frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x.$

46. $\frac{x+b}{a+b}+1=\frac{5a-x}{a+b}.$

47. $x-\frac{ax}{a+b}=\frac{ab}{a-b}-\frac{b^2x}{a^2-b^2}.$

48. $a-\frac{a+x}{b}=b-\frac{b+x}{a}.$

49. $(x+2a)(x+b)-(x+a)(x+2b)=(a+b)(x-2b).$

50. $a(x-a-2b)+c(x-c-2a)=b(b+2c-x).$ (*See Art.* 37.)

Find the values of x and y in terms of the other letters from the equations.

51. $px+qy=r.$
$qx+py=r.$

52. $\frac{x}{a+1}-\frac{y}{a-1}=2;\quad \frac{x-y}{2a}=\frac{x+y}{a^2+1}.$

53. $\frac{x}{a-c}+\frac{y}{b-c}=3;\quad 2(b+c)x-(a+c)y=4c(a-b).$

54. $(a-b)(x+2y)+(a+b)(2x+y)=a^3.$
$(a+b)(x+2y)+(a-b)(2x+y)=a^3+2ab^2.$

73. Examples on the Unitary Method, and Problems.

Ex. 1. *If x oranges can be bought for y shillings, what is the price in pence per score?*

$$x \text{ oranges cost } 12y \text{ pence;}$$

$$\therefore\quad 1 \text{ orange costs } \frac{12y}{x} \text{ pence;}$$

$$\therefore\quad 20 \text{ oranges cost } \frac{12y}{x}\times 20 \text{ pence.}$$

The price is therefore $\frac{240y}{x}$ pence per score.

Ex. 2. *Express in feet per second the speed of a train which, running at a uniform rate, travels x miles in y hours.*

In y hours the train travels x miles;

$\therefore$ in $y \times 60 \times 60$ seconds .... $x \times 1760 \times 3$ ft.;

$$\therefore \text{ in 1 second } \ldots\ldots\ldots\ldots \frac{x \times 1760 \times 3}{y \times 60 \times 60} \text{ ft.}$$

$$= \frac{x \times 22}{y \times 15}.$$

The speed is therefore $\dfrac{22x}{15y}$ ft. per second.

Ex. 3. *By selling a horse for £43 and one third as much as it cost, a man gained £5. What was the cost of the horse?*

Denote the cost price by £x, then the selling price was £$\left(43 + \dfrac{x}{3}\right)$;

$$\therefore \text{ gain in pounds} = \left(43 + \frac{x}{3}\right) - x,$$

and since the gain was £5,

$$\left(43 + \frac{x}{3}\right) - x = 5;$$

$$\therefore \; 43 + \frac{x}{3} - x = 5.$$

Multiplying each side by 3,

$$129 + x - 3x = 15,$$

whence $x = 57$, and the cost price was £57.

EXERCISE XXX.

1. Find the cost of (i) 60 apples at x pence per dozen; (ii) y apples at x pence per dozen.

2. If x apples cost y pence, (i) how many can be bought for a sovereign? (ii) what is the cost in shillings of z apples?

3. How many hours does it take to walk

   (i) x miles at the rate of 3 miles an hour?

   (ii) x miles at y miles an hour?

4. In x hours a man walks y miles:
   (i) At what rate does he walk in miles per hour?
   (ii) How many yards does he walk in 20 minutes?

5. If a train runs at the rate of x miles per hour what is its speed in yards per minute?

6. A man is walking at the uniform rate of m miles per hour: in how many seconds will he traverse y yards?

7. If x ounces of silver are worth y shillings, how many pennyweights of silver are worth z pence?

8. If a clock gains x seconds in 24 hours, in how many days will it be y minutes fast?

9. A rectangular court x ft. by y ft. is to be paved with bricks each a in. by b in. How many bricks are necessary?

10. It takes n bricks each a in. long to pave a square court whose side is x feet. What is the breadth of a brick?

11. A wall which is p feet long and q inches thick contains k bricks; each brick is a inches long, b inches broad and c inches thick: find the height of the wall.

12. If x men take y hours to mow a certain field, (i) how long will one man take to mow the field? (ii) how long will z men take?

13. If x men take 15 hours to mow a certain field and 10 men take y hours to mow the same field, what is the equation connecting x and y?

14. A room is l ft. long and b ft. broad.
   (i) How many yards of linoleum 6 ft. wide are required to cover the floor?
   (ii) Find the cost in shillings of the linoleum at the rate of z shillings per yard.

15. A room is l ft. long, b ft. broad and h ft. high.
   (i) Find the area of the walls in square yards.
   (ii) Find the length in yards of paper c ft. wide required to cover the walls.
   (iii) Find the cost in shillings of this paper at the rate of x pence for 12 yds.

16. If a men reap b acres in c days, how many acres will p men reap in q days?

17. If 10 horses eat x bushels of corn in 6 days and 15 horses eat y bushels in 2 days, find the equation connecting x and y.

18. What sum of money exceeds its sixth part by £10?

19. What is the sum of money such that its half exceeds the difference between its third and fourth parts by £10?

20. A post is a quarter of its length in the ground, a third of its length in water, and rises 10 feet above the water; what is its whole length?

21. There are two numbers whose sum is 125. Also $\frac{3}{4}$ of the one number exceeds $\frac{2}{3}$ of the other by 13. Find the numbers.

22. A number is divided into two parts; the difference between the parts is 5, and two-thirds of the smaller part is less than three-fourths of the larger part by 8. Find the number.

23. Divide 100 into 3 parts such that $\frac{2}{5}$ of the first part, $\frac{1}{3}$ of the second and $\frac{2}{9}$ of the third are all equal.

24. A man buys $\frac{2}{3}$ of an estate at £12 per acre, and the remainder at £20 per acre, and, by selling the whole at £18 per acre, he makes a profit of £500; find the size of the estate.

25. Find a number of three digits, each digit being greater by unity than that which follows it, so that the excess of this number above one-fourth of the number obtained by reversing the digits shall be 36 times the sum of the digits.

26. A bag contains a certain number of gold coins. Half its contents and half a sovereign more are removed. Half of what remains and half a sovereign more are then removed, leaving £5 in the bag. Find how much the bag contained originally.

27. A man sold a horse for £35 and half as much as it cost him, and by doing so gained £10. What was the original price of the horse?

28. A has 6 more marbles than B. If A were to give one-third of his marbles to B, B would then have 10 more than A. How many has each?

29. A had £50 more than B; A paid a third part of his money to B, and B paid back a fifth part of what he then had: the result was that B had £30 more than A. How much had each at first?

30. Find two numbers, whose sum is 620, which are such that twice the smaller number exceeds the greater number by one-seventh of the smaller number.

31. A party consists of men, women and boys; the men are one more than one-sixth of the whole party, the women two more than one-quarter of the whole, and the boys five more than one-half of the whole. How many are there altogether?

32. A man has a certain sum of money; he pays a bill for £1. 3*s*. 6*d*., and gives away half of what he has left; he then receives £2, and finds he has 6*d*. less than he had at first. Find how much he gave away?

33. A bill of £5 is paid partly in English shillings and partly in French francs. If the number of shillings were decreased by one-third, the number of francs would have to be increased by one-half. Find the number of each assuming that twenty-five francs are equal in value to £1.

34. Two men, A and B, play at cards; if B wins 12 shillings, he then has three-quarters of what A has; but if, instead, A wins a sovereign, he then has four times as much as B has. Find how much money each starts with.

35. A man walks 18 miles from A to B in 4 hours and 34 minutes, and does the return journey in 4 hours 50 minutes; walking 4 miles an hour on the level, 3 miles an hour uphill and 5 miles an hour downhill. Find how many miles there are on the level, how many up and how many down on his first journey.

36. A grocer buys a number of eggs at 6*s*. 6*d*. a hundred. He sells all but 69 of them at the rate of 11 for a shilling, and then finds that he has received 30*s*. more than he gave for the whole number. How many eggs did he buy?

37. Out of a flock of sheep a farmer sells a number so that he has $1\frac{1}{5}$ times as many left as he sold; 15 of these die, and he has then only half the original number. How many were in the flock?

38. One-fifth of the books of a library are out on a certain day. Of these one-fourth are fiction, one-sixth are history and the rest of them (280 in number) are of other kinds. How many books are still in the library?

CHAPTER IX.

EVOLUTION.

74. Square Roots and Cube Roots. In Arithmetic, the relation between the numbers 7 and 49 is expressed in two ways: 49 is called the *square of* 7, and 7 is called the *square root of* 49. The square root of 49 is denoted symbolically by $\sqrt{49}$, and the sign $\sqrt{}$ is called the **root** (or **radical**) **sign**.

Thus the equations $7^2 = 49$ and $\sqrt{49} = 7$ express one and the same relation.

Again, 8 is called the *cube of* 2 and 2 is called the *cube root of* 8, which is denoted by the expression $\sqrt[3]{8}$, so that the equations $2^3 = 8$ and $\sqrt[3]{8} = 2$ express the same relation.

The numbers 1, 4, 9, 16, ... are called **perfect squares**, and the numbers 1, 8, 27, 64, ... are called **perfect cubes**, these numbers being respectively the squares and cubes of the numbers 1, 2, 3, 4, ... of the natural scale. Thus it is *exceptional* for a number to be a perfect square or a perfect cube.

75. The n$^{\text{th}}$ root of a number. If a and n stand for given numbers, and if a natural number x can be found such that $x^n = a$, then a is called a **perfect n$^{\text{th}}$ power**, the number x is called the **n$^{\text{th}}$ root of a**, and is denoted by the expression $\sqrt[n]{a}$.

If a is a perfect n^{th} power, $\sqrt[n]{a}$ can be found by constructing a table of the n^{th} powers of the natural numbers, that is by finding the values of 1^n, 2^n, 3^n, ..., and proceeding until a number is found whose n^{th} power is a. This number is the value of $\sqrt[n]{a}$.

If a is not a perfect n^{th} power, instead of saying that a has no n^{th} root, we invent a new class of numbers called **irrational numbers** or **surds**, and we say that the n^{th} root of a is a number belonging to this class, which we denote by $\sqrt[n]{a}$.

Thus $\sqrt[3]{2}$ stands for a surd or irrational number whose cube is 2.

The process of finding a root of a given number is called **Evolution**.

76. Evolution and Involution are Inverse Operations. For by definition

$$(\sqrt[n]{a})^n = a = \sqrt[n]{a^n}.$$

Thus, to take the n^{th} root of a number a and raise the result to the n^{th} power, or to raise a number a to the n^{th} power and take the n^{th} root of the result is to leave the number a unaltered.

NOTE. The n^{th} root of the product abc is denoted by $\sqrt[n]{abc}$ or by $\sqrt[n]{(abc)}$, the line called a **vinculum** being equivalent to the bracket.

77. Theorems on Roots. The following theorems are fundamental in connection with roots of numbers. In every case it is assumed that a root exists which is a natural number.

Theorem 1. $\sqrt[n]{a}\,.\,\sqrt[n]{b} = \sqrt[n]{ab}.$

Proof. Let $\sqrt[n]{a} = x$ and $\sqrt[n]{b} = y$, then, by hypothesis, x and y stand for natural numbers, and

$$a = x^n, \quad b = y^n; \quad (Def.)$$

$$\begin{aligned} \therefore \quad \sqrt[n]{ab} &= \sqrt[n]{(x^n y^n)} \\ &= \sqrt[n]{(xy)^n} \qquad (Index\ Law) \\ &= xy \\ &= \sqrt[n]{a}\,.\,\sqrt[n]{b}. \end{aligned}$$

In the same way it can be shown that $\sqrt[n]{a}\,.\,\sqrt[n]{b}\,.\,\sqrt[n]{c} = \sqrt[n]{abc}.$

Theorem 2. $\sqrt[m]{\{\sqrt[n]{a}\}} = \sqrt[mn]{a} = \sqrt[n]{\{\sqrt[m]{a}\}}.$

Proof. Let $\sqrt[mn]{a} = x$, then, by hypothesis, x is a natural number, and

$$a = x^{mn};$$

$$\therefore \quad a = (x^m)^n; \qquad (Index\ Law)$$

$$\therefore \quad \sqrt[n]{a} = x^m; \qquad (Def.)$$

$$\therefore \quad \sqrt[m]{\{\sqrt[n]{a}\}} = \sqrt[m]{x^m} = x = \sqrt[mn]{a}.$$

In the same way it can be shown that $\sqrt[n]{\{\sqrt[m]{a}\}} = \sqrt[mn]{a}.$

Illustrations. (i) $\sqrt{4 \times 9 \times 49} = \sqrt{4} \times \sqrt{9} \times \sqrt{49} = 2\,.\,3\,.\,7$;

(ii) $\sqrt[6]{a} = \sqrt[3]{\{\sqrt{a}\}}$,

so that the sixth root of a number a may be found by finding

the square root of a and then finding the cube root of the result, thus

$$\text{(i)}\quad \sqrt[6]{64}=\sqrt[3]{8}=2.$$

$$\text{(ii)}\quad \sqrt{a^4b^6}=\sqrt{a^4}.\sqrt{b^6}=\sqrt{(a^2)^2}.\sqrt{(b^3)^2}=a^2b^3.$$

$$\text{(iii)}\quad \sqrt[3]{a^6b^9}=\sqrt[3]{a^6}.\sqrt[3]{b^9}=\sqrt[3]{(a^2)^3}.\sqrt[3]{(b^3)^3}=a^2b^3.$$

$$\text{(iv)}\quad \sqrt[n]{a^{2n}b^{n^2})}=\sqrt[n]{a^{2n}}.\sqrt[n]{b^{n^2}}=\sqrt[n]{(a^2)^n}.\sqrt[n]{(b^n)^n}=a^2b^n.$$

It will be seen from the above examples that if an expression is the product of factors such that (i) the index of the power to which each factor of the product is raised is an even number, then the square root of the expression can be found by dividing the index of every factor by 2; (ii) if the index of the power to which each factor of the product is raised is divisible by n, then the n^{th} root of the expression can be found by dividing the index of every factor by n. Thus if p, q, r are divisible by n,

$$\sqrt[n]{a^pb^qc^r}=a^{\frac{p}{n}}b^{\frac{q}{n}}c^{\frac{r}{n}}.$$

Ex. 1. *Express 3969 as a product of prime factors, and find the square root of 3969.*

We have $3969=3^4.7^2$; $\therefore\ \sqrt{3969}=\sqrt{3^4.7^2}=3^2.7=63.$

Ex. 2. *If the lengths of the sides of a triangle, measured in inches, are denoted by a, b, c, and if the area of the triangle is Δ square inches, it is found that*

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)},$$

where $2s=a+b+c$. Hence, find the area of a triangle whose sides are 70, 58 and 16 inches respectively.

$$s=\tfrac{1}{2}(a+b+c)=\tfrac{1}{2}(70+58+16)=72:$$

$$\therefore\ s-a=2,\quad s-b=14,\quad s-c=56;$$

$$\begin{aligned}\therefore\ \Delta&=\sqrt{s(s-a)(s-b)(s-c)}\\&=\sqrt{72.2.14.56}\\&=\sqrt{36.2.2.7.2.7.8}\\&=\sqrt{6^2.2^2.7^2.4^2}\\&=6.2.7.4\\&=336;\end{aligned}$$

$\therefore$ area of triangle $=336$ square inches.

The following points should be noticed:

1. In the last example, the first step in the process of finding

the square root of the product $72.2.14.56$ was to replace 72 by the product 36.2; the factor 36 was chosen as being a perfect square. Of course 72, 14, 56 might have been replaced each by a product of prime factors but this process would have been longer than the one employed.

2. Observe that whilst $\sqrt[3]{64}$ denotes *the cube root of* 64, the expression $3\sqrt{64}$ stands for 3 *times the square root of* 64, thus

$$\sqrt[3]{64}=4 \quad \text{and} \quad 3\sqrt{64}=3.8=24.$$

EXERCISE XXXI.

Simplify

1. $\sqrt[3]{x^6}$.
2. $3\sqrt{x^6}$.
3. $\sqrt[5]{x^{10}}$.
4. $5\sqrt{x^{10}}$.
5. $\sqrt{x^2y^4z^8}$.
6. $\sqrt{(x^2y^4)}\,.\,z^8$.
7. $\sqrt[3]{(x^6y^{18})}$.
8. $\sqrt{64x^{64}}$.
9. $\sqrt[3]{27x^{27}}$.
10. $\sqrt[n]{x^ny^{3n}}$.
11. $\sqrt[n]{(n^nx^{n^2})}$
12. $\sqrt[n]{(x^{3n}y^{n^3})}$.
13. $\sqrt[n]{(x^{n^2+n})}$.
14. $\sqrt[n]{(x^{n+1}\,.\,x^{n-1})}$.
15. $\sqrt{(x^n\div x^{n-2})}$.

By expressing each number concerned as a product of prime factors, find the values of

16. $\sqrt{196}$.
17. $\sqrt{441}$.
18. $\sqrt{484}$.
19. $\sqrt{10816}$.
20. $\sqrt[3]{1728}$.
21. $\sqrt[3]{3375}$.
22. $\sqrt[3]{21952}$.
23. $\sqrt[5]{32a^{10}}$.
24. $\sqrt[5]{625a^{25}}$.
25. $\sqrt[4]{(49)^2}$.
26. $\sqrt[6]{8^4}$.
27. $\sqrt[12]{(64^4a^{24})}$.

If $a=5$, $b=4$, $c=3$, find the values of

28. $\sqrt{15abc}$.
29. $\sqrt{14(b+c)(c+a)(a+b)}$.
30. $2\sqrt{a^2-b^2}+3\sqrt{a^2-c^2}+4\sqrt{b^2+c^2}$.
31. $\sqrt[3]{a^3+b^3+c^3}$.
32. $\sqrt[3]{2(b-c)^5(a-c)^5}$.
33. $\sqrt[3]{(b+c-a)^2(c+a-b)^2}$.

34. Find the least value of x which will render each of the following expressions a perfect square: (i) $200x$; (ii) $363x^3$; (iii) $432x^5$.

35. Find the least value of x which will render each of the following expressions a perfect cube: (i) $200x$; (ii) $32x^2$; (iii) $675x^4$.

36. Assuming the formula used in Ex. 2 of Art. 77, find the areas of the triangles whose sides, measured in inches, are as follows: (i) 26, 28, 30; (ii) 20, 34, 18; (iii) 34, 50, 56; (iv) 90, 80, 26.

37. If a, b, c, d respectively denote the number of units of length in the sides of a quadrilateral inscribed in a circle, and if Q is the number of units of area contained in the quadrilateral, it is found that

$$Q=\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $2s=a+b+c+d$.

Hence find the area of a quadrilateral inscribed in a circle, the sides of the quadrilateral being 41, 59, 69 and 81 inches respectively.

CHAPTER X.

GRAPHICAL REPRESENTATION OF NUMBER.

78. Number as a Measure. Except in the case of certain problems, a number has so far been regarded as a symbol occupying a definite place in the scale. We here consider the use of number in measurement.

If A and B denote two things of which it can be said that A is greater than, equal to or less than B, A and B are called *magnitudes of the same kind.* The greater of two magnitudes is said to *contain* the less.

To measure a magnitude is to compare it with some standard magnitude of the same kind called a **unit.** If it is found that the given magnitude contains the unit an exact number of times (say 4 times), this number (4) is called the **measure** of the given magnitude.

In ordinary language, anything which answers the question "How many?" or "How much?" is called a *Quantity.* In mathematics the word **Quantity** means "anything to which mathematical processes are applicable" (*Webster*), thus a *number* is often called a *quantity*, and that without any reference to measurement.

79. Addition and Subtraction of Lengths. Assuming that a straight line can be moved from any one position to any other position without altering its length, we say that the lengths of two straight lines are **equal** if one of the lines can be placed so as to coincide with the other.

Let AB and CD be two given straight lines and let OX be a straight line of unlimited length. It is assumed that along OX lengths OP, PQ can be set off which are respectively equal to the lengths AB and CD. If this is done, the length OQ is called the sum of the lengths AB and CD, and we write

$$OQ = OP + PQ = AB + CD, \text{ and } PQ = OQ - OP.$$

Here the signs $=$, $+$, $-$ are used to affirm in a convenient manner that *certain straight lines can be made to coincide.* Observe that OP, OQ, PQ do not stand for *numbers* and the signs have a meaning totally different from that assigned in Arts. 3, 14.

80. Measurement of Length. If AB and CD are two given straight lines, and if by setting off lengths along AB in succession each equal to CD, it is found that AB contains CD exactly 4 times, we say that AB is 4 times CD and we write $AB = 4CD$; if CD is taken as the unit of length, say one inch, then the number 4 is the measure of the length AB.

If AB does not contain an exact number of inches, we choose a smaller unit, say one-tenth of an inch or one-hundredth of an inch. If the length of AB lies between 725 and 735 hundredths of an inch, we say that the length of AB is 73 tenths of an inch *approximately, or correct to the nearest tenth.*

If lengths OP, PQ, containing respectively 3 and 2 units of length, are set off along a straight line OX, and *if OP, PQ stand for the measures of the lengths of the lines OP, PQ* (that is for the numbers 3 and 2), then the relations $OQ = OP + PQ$, $PQ = OQ - OP$ are the same as $5 = 3 + 2$ and $2 = 5 - 3$, and the signs $=$, $+$, $-$ have their usual meanings.

81. Representation of Numbers by Points. If, starting from a given point O in a straight line OX (which may be produced to any length), an indefinite number of equal lengths are set off in succession along the line, and if the points of division are marked

$$1, 2, 3, 4, \ldots\ldots$$

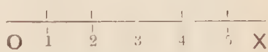

Fig. 2

in order, we obtain points which, *by their position*, represent the numbers of the natural scale in the following respects:

(i) for every number, there is one point and only one;

(ii) the points occur in the order in which the corresponding numbers stand on the scale.

82. Representation of Pairs of Numbers by Points. Draw two straight lines OX, OY, each of indefinite length, and at right angles to one another (Fig. 3). Choose some convenient unit of length, say one tenth of an inch. Along OX set off a length ON to contain 6 units, along OY set off a length OM to contain 4 units; through N draw NP parallel to OY and through M draw MP parallel to OX. The point P (Fig. 3) where MP, NP intersect will be said to *represent* by its position the pair of numbers (6, 4), and will be called the point (6, 4).

In Fig. 3 the points Q, R represent the pairs of numbers (8, 8), (4, 6) respectively.

It will be seen that points constructed in the manner just described, represent pairs of numbers in this respect :—

For every pair of numbers there is one point and only one.

Observe that the numbers forming a "pair" are written in a definite order; thus the pair (4, 6) is different from the pair (6, 4), and these pairs correspond to different points (namely to R and P in Fig. 3).

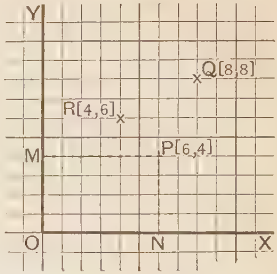

FIG. 3.

Further, it can be shown that *to every point in the plane OXY there corresponds one pair of numbers and one only*, but in general the numbers are of a class different from the natural numbers.

The point O is called the **origin**. The lines OX, OY are called the **axes (of coordinates)**; of these OX is the **axis of x** and OY the **axis of y**.

In Fig. 3, ON and NP (which is equal to OM) are called the **coordinates** of P; of these ON is called the **abscissa** of P and NP the **ordinate** of P.

It is not essential to the method described in this article of representing pairs of numbers by points that the unit of length chosen for measurement along OY should be the same as that for measurement along OX. If different units are to be used for measurements along the two axes this will be distinctly stated.

To **plot** a point, whose coordinates are given, is to mark its position in a diagram.

In graphical work we shall take "the length of the line AB" to mean "the measure of the length AB," so that the statement $AB = 4$ means that the line AB contains 4 units of length.

The **unit of area** is the area of a square whose side is the unit of length; thus if the unit of length is one-tenth of an inch, the unit of area is the area of the shaded square in Fig. 4. Now one square inch contains 10×10 or 100 of these small squares, hence in this case the unit of area is one-hundredth of a square inch.

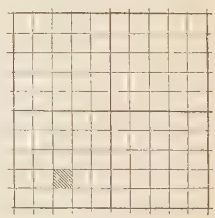
FIG. 4.

83. Various Facts in Geometry. A knowledge of the following facts in geometry is required:

1. Any side of a parallelogram may be called the *base*. The *altitude* or *height* of a parallelogram is the length of the perpendicular drawn from any point in the base to the opposite side.

If b, h are the numbers of units of length in the base and height of a parallelogram (Fig. 5),

$$\text{area of parallelogram} = (bh) \text{ units of area.}$$

Fig. 5.

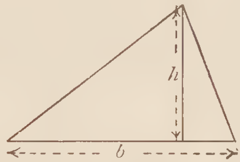

Fig. 6.

2. If b is the base* and h the altitude of a triangle (Fig. 6), then

$$\text{area of triangle} = \tfrac{1}{2}bh \text{ units of area.}$$

3. If a, b are the lengths of the parallel sides of a trapezium and if h is the perpendicular distance between these sides (Fig. 7),

$$\text{area of trapezium} = \tfrac{1}{2}(a+b)h.$$

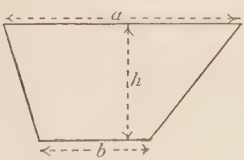

Fig. 7.

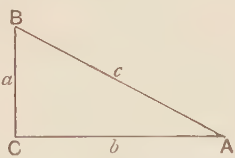

Fig. 8.

4. In the triangle ABC, a, b, c denote the lengths of the sides BC, CA, AB (Fig. 8).

If ABC is a right-angled triangle, C being the right angle, then $c^2 = a^2 + b^2$.

Conversely if, in the triangle ABC, $c^2 = a^2 + b^2$, then the angle C is a right angle.

---

*To say that b is the base of a triangle means that b represents the number of units of length in the base.

84. Examples. In the following, the unit of length is taken as 0·1 in.

Ex. 1. *Plot the point (6, 8). If this point is denoted by P and PN is the perpendicular from P to OX, find* (i) *the area of the triangle OPN*; (ii) *the length of OP.*

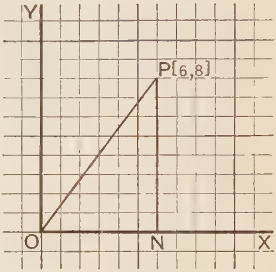

FIG. 9.

(i) Area of $\triangle OPN$

$= \frac{1}{2} ON . NP$

$= \frac{1}{2} . 6 . 8$ units of area

$= 24$ units of area.

Thus the area of $\triangle OPN$ is 24 hundredths of a square inch, or 0·24 sq. in.

(ii) $\because \angle N$ is a right angle,

$\therefore\ OP^2 = ON^2 + NP^2$

$= 6^2 + 8^2 = 2^2(3^2 + 4^2) = 2^2 . 5^2$;

$\therefore\ OP = 2 . 5 = 10.$

The length of OP is therefore 10 tenths of an inch, or 1 inch.

Ex. 2. (i) *Plot the points (3, 2), (15, 7).* (ii) *Find the distance between these points.* (iii) *If the points are denoted by P and Q and PM, QN are drawn perpendicular to OX, find the area of the trapezium PMNQ.*

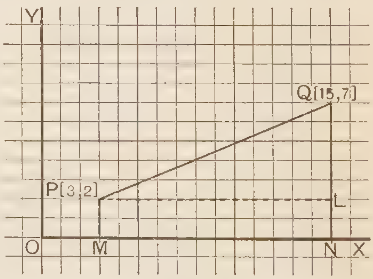

FIG. 10.

(ii) Draw $PL \perp QN$, then

$PL = MN = ON - OM$

$= 15 - 3 = 12,$

$LQ = NQ - NL$

$= NQ - MP = 7 - 2 = 5,$

and $\because \angle PLQ$ is a right angle,

$PQ^2 = PL^2 + LQ^2$

$= 12^2 + 5^2 = 169 = 13^2$;

$\therefore\ PQ = 13$;

the distance PQ is therefore 13 tenths of an inch, or 1·3 inches.

(iii) Area of $PMNQ = \frac{1}{2}(MP + NQ) MN$

$= \frac{1}{2}(2 + 7)(15 - 3) = \frac{1}{2} . 9 . 12 = 54$;

$\therefore$ $PMNQ$ contains 54 units of area, or 0·54 sq. in.

Ex. 3. *Plot the points (3, 2), (15, 6), (10, 10) and find the area of the triangle of which these are the vertices.*

If A, B, C are the points in question, draw $AM \parallel OX$, BM, $CN \perp AM$, then

$\triangle ABC = \triangle ANC + \text{fig. } CNMB - \triangle AMB$

$= \frac{1}{2} AN . NC + \frac{1}{2}(NC + MB) NM - \frac{1}{2} AM . MB$

$= \frac{1}{2} . 7 . 8 + \frac{1}{2}(8 + 4) 5 - \frac{1}{2} . 12 . 4$

$= 28 + 30 - 24$

$= 34$;

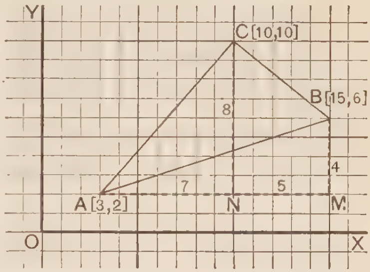

Fig. 11.

$\therefore$ area of $\triangle ABC = 34$ units of area $= 0{\cdot}34$ sq. in.

Ex. 4. *Plot the points (11, 4), (15, 8), (7, 16), (3, 7) and find the area of the quadrilateral of which these are the vertices.*

Construct the rectangle $LMNR$ by drawing RAL, $NCM \parallel OX$ and LBM, $RDN \parallel OY$. Then

area of $ABCD$

$= LMNR - \triangle ALB - \triangle BMC - \triangle CND - \triangle DRA$

$= 12 . 12 - \frac{1}{2} . 4 . 4 - \frac{1}{2} . 8 . 8 - \frac{1}{2} . 4 . 9 - \frac{1}{2} . 3 . 8$

$= 144 - 8 - 32 - 18 - 12$

$= 74$;

$\therefore$ $ABCD$ contains 74 units of area or $0{\cdot}74$ sq. in.

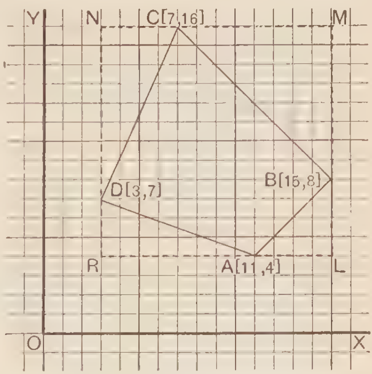

Fig. 12.

Second method. By drawing parallels to OX through B and D, and parallels to OY through A and C, divide $ABCD$ into four triangles and a rectangle.

EXERCISE XXXII.

1. Plot the following sets of points (each set in a separate diagram). Draw the figures whose vertices are the points of each set. What kind of quadrilateral is each figure?
   (i) (1, 1), (9, 3), (11, 8), (3, 6); (ii) (9, 1), (11, 5), (3, 9), (1, 5); (iii) (1, 1), (8, 4), (11, 11), (4, 8); (iv) (6, 2), (12, 5), (9, 11), (3, 8); (v) (5, 3), (12, 6), (9, 13), (2, 10).

2. The points in the following sets are the vertices of rectangles; draw the rectangles and calculate their areas.
   (i) (3, 2), (11, 2), (11, 7), (3, 7); (ii) (2, 1), (6, 1), (6, 8), (2, 8).

3. Draw the triangles whose vertices are the points given in (i)–(iv). Calculate the area of each triangle.
   (i) (3, 3), (11, 3), (9, 7); (ii) (1, 2), (5, 2), (10, 8); (iii) (2, 2), (9, 7), (2, 8); (iv) (2, 6), (8, 1), (2, 9).

4. Draw the trapezium whose vertices are the points given in (i)–(iv). Calculate the area of each trapezium.
   (i) (2, 1), (11, 1), (7, 6), (4, 6); (ii) (1, 2), (6, 2), (12, 7), (9, 7); (iii) (3, 10), (9, 3), (9, 15), (3, 13); (iv) (2, 2), (8, 1), (8, 8), (2, 5).

5. Draw triangles with one vertex at the origin, the other vertices being the points given in (i)–(iii). Calculate the area of each triangle.
   (i) (8, 5), (2, 9); (ii) (6, 2), (10, 9); (iii) (10, 6), (6, 8).

6. Draw the triangles whose vertices are given in (i)–(iv). Calculate the areas of the triangles.
   (i) (2, 3), (10, 6), (6, 9); (ii) (6, 1), (9, 3), (2, 7); (iii) (9, 2), (7, 7), (1, 5); (iv) (2, 3), (10, 7), (14, 15).

7. Draw the quadrilaterals whose vertices are given in (i)–(vii). Calculate the areas of the quadrilaterals.
   (i) (3, 2), (9, 2), (11, 6), (3, 10); (ii) (5, 1), (15, 8), (7, 13), (3, 8); (iii) (13, 1), (21, 8), (7, 15), (3, 4); (iv) (5, 1), (11, 8), (7, 16), (3, 6); (v) (10, 2), (16, 10), (7, 16), (5, 8); (vi) (2, 2), (16, 5), (11, 13), (5, 10); (vii) (5, 3), (17, 15), (9, 17), (3, 5).

8. Calculate the distance of each of the following points from the origin:
   (9, 12), (10, 24), (16, 30), (14, 48).

9. Plot the pairs of points given in (i)-(iv). Calculate the distance between the points of each pair.

(i) (2, 6), (5, 2); (ii) (3, 7), (11, 1);
(iii) (2, 1), (14, 6); (iv) (2, 2), (10, 17).

10. Show that the points (16, 63), (25, 60), (39, 52), (52, 39), (60, 25), (63, 16) lie on a circle whose centre is at the origin. Find the radius of the circle.

11. Plot the points (2, 2), (3, 5), (4, 6), (10, 6), (11, 5), (12, 2). Prove that they all lie on a circle whose centre is (7, 2). Find the radius of the circle.

12. If (x, y) is a point on a circle whose centre is at the origin and whose radius is r, what is the equation connecting x, y and r?

13. If (x, y) is a point on a circle whose centre is (a, b) and whose radius is r, what is the equation connecting x, y, a, b, r?

14. Prove that the straight line joining the origin to the point (24, 7) subtends a right angle at the point (12, 16).

15. Prove that the triangle whose vertices are the points (4, 2), (12, 8), (1, 6) is a right-angled triangle.

85. The Linear Equation. If x and y are variables, such equations as $x=4$, $2x=3y$, $3x+y=20$, which are of the *first degree* in one or both of the variables, are called **linear** equations.

We shall plot a number of points whose coordinates are connected by a linear equation.

Take as an example the equation

$$y=2x+4.$$

Giving to x the values 1, 2, 3, 4, 5, ..., we find the corresponding values of y. Arranging the results as follows,

x	1	2	3	4	5
y	6	8	10	12	14

plot the points (1, 6), (2, 8), (3, 10), (4, 12), (5, 14), and observe that they appear to lie on a straight line (AB in Fig. 13).

That this is the case may be verified by applying a ruler to the diagram.

Next take any other solution of $y=2x+4$, say, (8, 20). Plot the point (8, 20), and observe that it lies on AB.

Again, take any point in AB whose coordinates we are able to express by means of numbers, for instance the point R, whose coordinates are (7, 18), and observe that if $x=7$ and $y=18$, the equation $y=2x+4$ is satisfied.

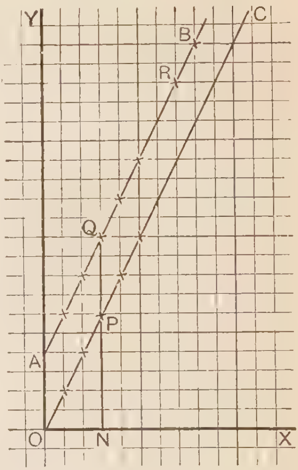

Fig. 13.

In this way we can satisfy ourselves that

(i) every point whose coordinates satisfy $y=2x+4$ lies on the straight line AB (if produced far enough);

(ii) if the coordinates of a point do not satisfy the equation, then the point does not lie on the straight line;

(iii) the coordinates of every point in the line, which we are able to express by means of numbers, satisfy the equation.

We have thus obtained experimentally a geometrical interpretation of the equation $y=2x+4$. This equation has been found to correspond to a certain straight line AB, which is called the **graph** of the equation, whilst $y=2x+4$ is called the **equation to the straight line AB**.

Next consider the equation $y=2x$. We tabulate a number of solutions

x	1	2	3	4	5
y	2	4	6	8	10

and plot the points (1, 2), (2, 4), (3, 6), (4, 8), (5, 10) (Fig. 13). By applying a ruler to the diagram, we can verify the fact that these points lie on a straight line (OC in Fig. 13) *which passes through the origin.* We can also verify that the line OC, produced if necessary, contains any point whose coordinates satisfy $y=2x$. Hence *the equation $y=2x$, which contains no constant term, represents a straight line through the origin.*

Note also that if P is any point in OC whose ordinate NP meets AB in Q, then $PQ=QN-NP=4$, so that PQ is constant

for different positions of P in OC, hence AB is parallel to OC. Thus *the equations $y=2x$ and $y=2x+4$, which differ only as regards the constant term, represent parallel straight lines.*

86. Lines parallel to the Axes. Plot a number of points which have the same abscissa (4). Such points are (4, 1), (4, 2), (4, 3)....

The coordinates of each of these points satisfy the equation $x=4$, *which asserts that x always stands for 4, whilst y may have any value.* All these points lie on the line AB (Fig. 14) which is parallel to OY, and the abscissa of every point on AB is 4, hence we say that $x=4$ is the equation to the straight line AB.

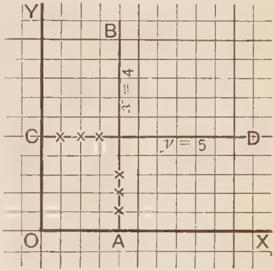

FIG. 14.

Again, the straight line CD (Fig. 14) contains all the points (1, 5), (2, 5), (3, 5)..., whose coordinates satisfy $y=5$, and the ordinate of every point in CD is 5, hence $y=5$ is the equation to the line CD which is parallel to OX.

87. Summary. Experiments (like those described in Arts. 84, 85) in plotting points whose coordinates satisfy a linear equation point to the truth of the following statements:

1. Every linear equation in x and y represents a certain straight line in the following respects: (i) every point whose coordinates satisfy the equation lies on the line; (ii) if the coordinates of a certain point do not satisfy the equation, then the point is not on the line; (iii) the coordinates of every point on the line, which we can express by means of numbers, satisfy the equation.

2. Equations like $y=3x$, $2x=5y$, which have no constant term, represent straight lines through the origin.

3. Equations like $2x=3y$, $2x=3y+4$, $2x=3y-6$, which differ only in respect to their constant terms, represent parallel straight lines.

4. The equation $x=a$ represents a straight line parallel to the axis of y, and $y=b$ represents a straight line parallel to the axis of x.

The truth of these facts will now be assumed.

88. Graphical Solution of Simultaneous Equations. To solve graphically the equations

$$x + y = 17, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$
$$3x - 2y = 6, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

we draw the graphs of (α) and (β), which are straight lines. To draw the graph of (α) we find any two points on it. *That the drawing may be as accurate as possible, we choose two points as far apart as the size of the paper permits.*

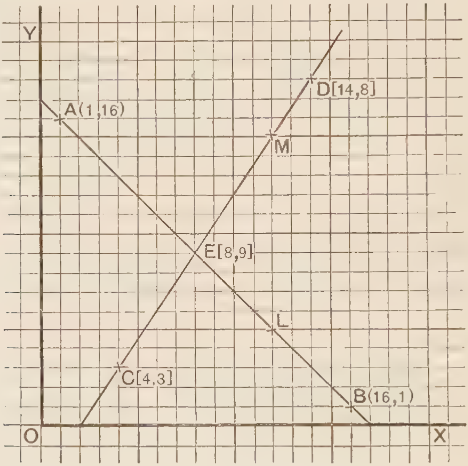

FIG. 15.

In (α), when $x=1$, $y=16$, and when $x=16$, $y=1$. Plot the points (1, 16), (16, 1) and join them; this is the line AB, which is the graph of (α).

In (β), when $x=4$, $y=3$, and when $x=14$, $y=18$. Plot the points (4, 3), (14, 18) and join them; this is the graph CD of (β).

The lines AB, CD meet in a point E, whose coordinates (if they can be expressed by numbers) satisfy both (α) and (β), for E lies on both the lines. The coordinates of E are (8, 9), hence (8, 9) is the solution of the given equations.

89. Graph of a Linear Function of x. Any expression containing a variable x which has a definite value for every

value of x is called a **function of x**. Thus $2x^2+3x-4$ and $5x+6$ are functions of x. Such expressions as $5x+6$, $3x$, $\frac{3x-6}{2}$, which contain no power of x higher than the first, are called **linear functions of x**. We can represent the values which $\frac{3x-6}{2}$ assumes for different values of x graphically as follows:

Let $\frac{3x-6}{2}=y$; draw the graph of the equation $\frac{3x-6}{2}=y$; this equation is the same as $3x-2y=6$, whose graph is the straight line CD in Fig. 15.

We can read off from the diagram the value of $\frac{3x-6}{2}$ (that is of y) for different values of x. To read off the value of $\frac{3x-6}{2}$ when $x=8$, we look for the point in CD whose abscissa is 8; this is the point E, whose ordinate is 9, hence 9 is the value of $\frac{3x-6}{2}$ when $x=8$. The line CD is called the graph of the equation $y=\frac{3x-6}{2}$, or **the graph of the function** $\frac{3x-6}{2}$.

Ex. *Draw graphs of the functions* $17-x$ *and* $\frac{3x-6}{2}$. *Read off from the diagram the values of these* (i) *when* $x=12$; (ii) *when the functions have equal values.*

The graphs are those of the equations $y=17-x$, $y=\frac{3x-6}{2}$, that is of (α), (β) of Art. 88. The graphs are therefore the straight lines AB, CD of Fig. 15.

(i) The points on AB, CD whose abscissae are equal to 12 are the points L, M, and the ordinates of L, M are 5 and 15 respectively;

$$\therefore \text{ when } x=12,\ 17-x=5 \text{ and } \frac{3x-6}{2}=15.$$

(ii) The ordinate of E, the intersection of the lines, is 9; this is therefore the value of both $17-x$ and $\frac{3x-6}{2}$ when these expressions are equal.

EXERCISE XXXIII.

1. For the equations (i)–(ix), (*a*) Plot the points obtained by giving to x the values 2, 8, 12, 14, making a separate diagram for each equation.

(*b*) Verify that the four points of each set lie on a straight line.

(*c*) Plot the point on each line whose abscissa is 6 and the points whose ordinate is 15. Verify that the coordinates of each of these points satisfy the equation.

(i) $x=y$; (ii) $3x=2y$; (iii) $x+y=19$;
(iv) $y-x=3$; (v) $y-2x=3$; (vi) $2y-x=2$;
(vii) $x+2y=32$; (viii) $2y-3x=6$; (ix) $3x+2y=42$.

What is the equation to the path of a point which moves as described in Ex. 2–5? In each case draw the graph.

2. The abscissa of the point is constant and equal to 6.

3. The ordinate is constant and equal to 8.

4. The abscissa is always equal to the ordinate.

5. The abscissa is always three times the ordinate.

6. A point moves so that the sum of its distances from two given straight lines OX, OY which are at right angles is always 15 units of length. What is the equation to the path of the point? Draw the graph of the equation.

7. Draw the lines whose equations referred to the same axes are $x+y=4$, $x+y=8$, $x-y=2$, $y-x=2$. What kind of figure is bounded by the lines? What is the area of the figure?

Solve graphically the following simultaneous equations:

8. $y-x=2$, $2x-y=5$.

9. $y=19-x$, $3x-y=21$.

10. $4x=3y$, $x=2(y-10)$.

11. $x=y+8$, $3x=4(y+3)$.

12. $x=3y-23$, $y=41-3x$.

In each of the examples 13–15 show graphically that the three equations are satisfied by the same values of x and y, and find these values.

13. $y-x=1$, $2x-y=11$, $3x-2y=10$.

14. $x+y=28$, $4x-3y=7$, $y=3x-24$.

15. $5x=6y$, $2x-y=14$, $x+y=22$.

In each of the examples 16–18 draw the graphs of the two given functions, and find the value of each function when the two have the same value.

16. $35-2x$; $\frac{1}{3}(x+21)$. **17.** $3x-8$; $\frac{1}{4}(3x+4)$. **18.** $\frac{1}{2}(10-x)$; $\frac{1}{3}(12-x)$.

90. Applications. The following examples show how graphs may be employed to show at a glance the way in which a variable quantity changes.

Ex. 1. *The average yearly price of £100 stock of a certain Railway Company, for ten years, is shown in the following table; illustrate graphically.*

Year - - - -	1897	98	99	00	01	02	03	04	05	06	07
Average price of stock	88	95	85	64	66	58	52	54	58	50	42

Take two thick lines OX, OY on squared paper.

Let lengths measured along OX represent time and lengths measured along OY represent price of stock.

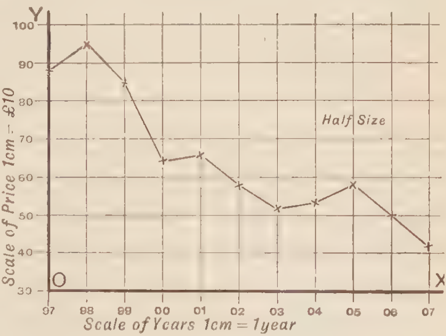

FIG. 16.

Next select suitable units of length to represent time and price. Choosing these as described in Fig. 16, mark points '98, '99, ... along OX, distant 1, 2, ... centimetres from O. Mark points 40, 50, ... along OY, distant 1, 2, ... cm. from O. The point O is the point '97 on the scale of years and the point 30 on the scale of price. Along the ordinates through '97, '98, ... set off lengths to represent prices of 88, 95, ...

Observe that the *price of a stock at any time is not connected with the time by any definite law*; thus if we were not told that in 1900 the price was 64, the rest of the information would not enable us in the least to say what the price was in 1900.

In such cases, it is usual to complete the diagram by joining consecutive points in the figure by straight lines.

Ex. 2. *A rifle sighted at 1000 yards is fired from a point O close to the ground. The height of the bullet above the line of sight (which is horizontal) is given in the following table:*

Horizontal distance from O in yards	200	300	400	500	600	700	800	900
Height of bullet in feet - - - - -	14·1	19·6	23·6	25·8	26·2	24·1	19·3	11·3

Represent the path of the bullet on squared paper, and find (i) *the approximate height of the bullet when its horizontal distance from O is 100 yards;* (ii) *the part of the range where a man 6 feet high may safely stand.*

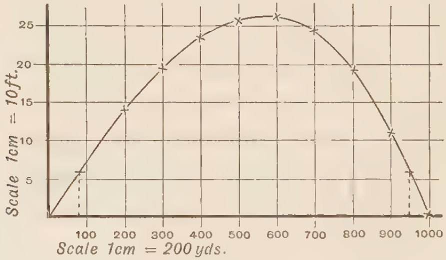

Fig. 17.

Choosing the scales for measuring horizontal and vertical distances as described in Fig. 17, we set off lengths along the verticals through the points marked 200, 300, ... to represent heights of 14·1 ft., 19·6 ft., ... We have now a number of points on the required graph.

Next observe the following points in connection with the motion of the bullet:

(i) The height of the bullet is connected with its distance from O, measured horizontally, by a definite law (although we cannot at present state the law mathematically).

(ii) As the height changes from one value to another (say from 300 to 400 feet), *it passes through every intermediate value* and does not alter by sudden jumps; it is therefore said to **vary continuously**.

(iii) The direction of the bullet also varies continuously. We therefore draw what is called a **smooth curve** through the points which have been plotted, in order to represent the intermediate positions of the bullet.

The height of the bullet at 100 yards from O is given by the intersection of the graph with the vertical through the point 100, and is about 7·5 ft.

The dotted lines in Fig. 17 are the ordinates of points on the graph which represent positions of the bullet 6 ft. above the ground. The horizontal line between the dotted lines therefore represents that part of the range where the man may safely stand. Its length is about 870 yards.

91. Graph of a Physical Quantity. Ex. 2 of Art. 90 is an instance of the graphical representation of a physical quantity. The method is as follows: certain points are found on the graph as the results of certain observations. A well-rounded curve is drawn to pass through or *near to* each point. If the points have been correctly plotted and such a curve has been drawn through or near to most of the points, and if it is found that some of the points are at a considerable distance from the curve, then it is probable that the corresponding observations are faulty.

The process by which in Ex. 2 of Art. 90 we were able to find the approximate height of the bullet at 100 yards is called **interpolation**.

EXERCISE XXXIV.

1. The average yearly price of wheat and barley per quarter from 1890 to 1900 is shown in the following table. Draw graphs referred to the same axes to show the variation in price of wheat and of barley.

Year - - - -	1890	1891	1892	1893	1894	1895
Price of wheat	31*s.* 11*d.*	37*s.*	30*s.* 11*d.*	26*s.* 4*d.*	22*s.* 10*d.*	23*s.* 1*d.*
Price of barley	28*s.* 8*d.*	28*s.* 2*d.*	26*s.* 2*d.*	25*s.* 7*d.*	24*s.* 6*d.*	21*s.* 11*d.*
Year - - - -	1896	1897	1898	1899	1900	
Price of wheat	26*s.* 2*d.*	30*s.* 2*d.*	34*s.*	25*s.* 2*d.*	26*s.* 11*d.*	
Price of barley	22*s.* 11*d.*	23*s.* 6*d.*	27*s.* 2*d.*	25*s.* 6*d.*	24*s.* 11*d.*	

Along OX take 2 small divisions of the paper to represent 1 year and along OY take 4 divisions to represent 1 shilling. Mark the origin as 1890 on the scale of years, and as 20*s.* on the scale of price.

2. A rifle sighted at 800 yards is fired from a support 5 feet above the ground (which is horizontal) at an object 5 feet above the ground. The height of the bullet above the line of sight is given in the table.

Horizontal distance from firing point in yards	200	300	400	500	600	700
Height of bullet in feet	9·3	12·3	13·9	13·8	11·7	7·2

Represent the path of the bullet on squared paper, and find (i) the approximate height of the bullet above the ground at 100 yds.; (ii) the part of the range which is safe for cavalry, taking the height of cavalry at 8 ft. 6 in., estimating the length of this part of the range. Show the height of the support in the diagram.

3. A rifle sighted at 1200 yards is fired from a point O close to the ground. The height of the bullet above the line of sight (which is horizontal) is given in the table.

Horizontal distance from O in yards	200	300	400	500	700	800	900	1000	1100
Height of bullet in feet - - - - -	19·8	28·3	35·2	40·4	44·4	42·4	37·5	28·8	16·6

Represent on squared paper the path of the bullet, and estimate its height at 100 yds. and at 600 yds. from O.

4. The following table taken from the *Daily Mail Year-Book* shows the percentage of the unemployed members of certain trade unions engaged in various trades:

Year - - -	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906
Engineering	3·8	3·8	2·5	2·8	3·8	5·1	4·9	7·0	5·1	3·0
Shipbuilding	7·6	4·7	2·3	2·5	3·7	8·2	12·0	14·0	11·9	7·6
Building - -	1·6	1·3	1·5	2·5	3·7	4·3	4·9	7·7	8·3	7·2

Draw three graphs referred to the same axes to show the variation of unemployment in the three trades. Indicate

that for the building trade by a dotted line. Take 10 small divisions of the paper along OX to represent 1 year and 10 divisions along OY to represent 1 per cent. Mark the origin 97 and 0 on the two scales.

5. A stone is thrown horizontally from the top of a cliff with a velocity of 100 feet per second. If x and y are the horizontal and vertical distances, measured in feet, travelled by the stone during the first t seconds of its motion, it can be shown that, neglecting the resistance of the air,

$$x=100t, \quad y=16t^2.$$

Tabulate the values of x and y for the values 1, 2, 3, 4 ... 10 of t, and by plotting these values obtain a representation of the path of the stone. Take OX horizontal and OY vertically downwards; scale for x, 100 ft. to 2 cm.; scale for y, 100 ft. to 1 cm.

6. In an experiment on the reflection of light by a concave spherical mirror, the distances u, v of the object and image from the mirror, measured in centimetres, were found to be:

u	6	7·5	10	14	26	40	60	90
v	30	15	10	7·8	6·2	5·7	5·5	5·3

Find, from a graph,

(i) the approximate values of v, when $u=11$, 20, 35;
(ii) the approximate values of u, when $v=8$ and 6.

7. A body is acted upon by a force which alters. When the body has passed through a distance of x feet from the starting point the force is equal to a weight of F lbs., and corresponding values of x and F are as follows:

x	0	0·1	0·2	0·3	0·4	0·6	0·8	1·0
F	20	21	21	20	19	18	9	0

How far has the body travelled when F has the values 15, 12, 6, 3?

8. In an experiment on the compressibility of a certain gas under pressure the corresponding values of the pressure (p millimetres of mercury) and the volume (v cubic centimetres) were found to be:

p	750	1125	1500	2250	675	500	375	300
v	4·5	3	2·25	1·5	5	6·75	9	11·25

Find, from a graph,

(i) the approximate values of p, when $v = 2·5$, $7·5$;

(ii) the approximate values of v, when $p = 1250$, 450, 900.

9. A combination is formed with a fixed lens and another lens of focal length f inches placed at a certain fixed distance from the fixed lens. The focal length of the combination is F inches. Corresponding values of f and F were found to be those in the table below:

f	3	4	5	10	15	20	30	45
F	3·8	4·4	5	6·7	7·5	8	8·6	9

It is required to make two combinations, one of focal length 6 and another of focal length 7. What values must be chosen for f?

10. The current (C) in amperes passing through a certain resistance, and the electro-motive force (E) which drove it, in a certain experiment were found to be:

C	7	5	3	2	1·5	1	·5	·2
E	120	85	50	35	26	17	9	3·5

Draw a smooth curve to pass as nearly as possible through the points corresponding to these values. State for which values of C you think the value of E is (i) too high, (ii) too low, (iii) nearly correct.

PART II. ZERO AND NEGATIVE NUMBERS

CHAPTER XI.

THE EXTENDED SCALE.

92. A New Class of Numbers. If letters are used to represent natural numbers only, no value of x can be found to satisfy the equation $x+b=a$ unless a is greater than b. We therefore *invent* a new class of numbers, in order that this equation may always have a solution.

93. Zero. As 1 is the first symbol in the natural scale, we cannot count back beyond it. We therefore *invent a new symbol* 0, called **zero** or **nought**, to place before 1. Zero is then the symbol reached by starting with any number a on the scale and counting a numbers backward along the scale; or a is the number reached by starting at 0 and counting a places forward. In other words, 0 *is defined by either of the equations*

$$a-a=0 \text{ or } 0+a=a.$$

94. Negative Numbers. In order that the equation $x+b=a$ may have a solution when b is greater than a, we extend the scale to the left of zero by *inventing* the symbols -1, -2, -3, etc., and placing -1 before 0, -2 before -1, -3 before -2, etc. The scale thus extended is

$$\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots.$$

The symbols -1, -2, -3, etc., are called **negative numbers** in order to distinguish them from 1, 2, 3, etc., which are called **positive numbers**. In order to mark this distinction, the latter are often written $+1$, $+2$, $+3$, etc., and further, the numbers are often enclosed in brackets, thus $(+2)$, (-3).

The following points should be noticed :

(1) The sign $-$ as used here has nothing whatever to do with subtraction : "minus" is part of the name of one of the new numbers, and means that the number in question is to be reached by counting backwards from 0.

(2) The scale extended as above is a collection of symbols *arranged in a definite order.* This is called the **ordinal** property of the scale, and the new numbers ... -3, -2, -1, 0 have this in common with the numbers 1, 2, 3, ... that they all form part of an ordinal system. This we consider to be a sufficient reason for calling the new symbols *numbers*, although other reasons for this will appear later on.

(3) *The numbers 1, 2, 3, etc., may be used to denote the number of things in a group.* This is called the **cardinal** property of the natural numbers. The negative numbers have no cardinal property, and to speak of (-3) cows or of (-3) steps is to talk nonsense.

(4) If x stands for a natural number, the numbers x and $(-x)$ are the same number of places to the right and left of zero on the scale : this is what is meant by the *symmetry* of the scale with regard to zero.

(5) The scale extended as above has *no first number* and *no last number*, and can be used for counting as many places backwards or forwards as we please.

95. Definitions of Equality and Inequality. If a and b stand for any numbers, whether positive, zero, or negative,

(1) a is said to be equal to b if a and b stand for the same number.

(2) a is said to be greater than b, or less than b according as a follows b, or as a precedes b, on the scale.

Thus $0 > -2$ and $-5 < -3$. It is important to observe that if $a > b$, then $-a < -b$, for if $a > b$, then by definition, a follows b and therefore also $-a$ precedes $-b$ on the scale.

The following important example illustrates the symmetry of the scale with regard to the symbol 0 :

Ex. 1. *If* $-x = a$, *then* $x = -a$.

For a and $-a$, and also x and $-x$ are the same number of places from zero on the scale. If then $-x$ is the same number as a, it follows that x is the same number as $-a$.

96. Addition and Subtraction. We can now extend the definitions of addition and subtraction contained in Arts. 3 and 14 as follows:

If a stands for any number which may be positive, zero or negative, and if b stands for a *positive number*, then

(1) $a+b$ stands for the bth number after a on the scale;

(2) $a-b$ denotes the number to which if b is added, the result is a.

Hence subtraction is still the inverse of addition and the number denoted by $a-b$ occupies the bth place before a on the scale.

Thus, to subtract 5 from 3, start from 3 and count 5 symbols backwards on the scale, $(2, 1, 0, -1, -2)$; the number reached is (-2), so that $3-5=(-2)$.

To subtract 2 from 0, start with 0 and count 2 places backwards, thus $0-2=(-2)$.

Observe that on the left-hand sides of these equations the sign $-$ denotes subtraction, and on the right-hand sides "minus" is part of the name of a new number.

Ex. 1. *If b, c, x are positive numbers and $x=c-b$, prove that* $-x=+b-c$.

Since x and $-x$ are the same number of places from zero on the scale, if x is reached by counting c forward and b backward along the scale, then $-x$ is reached by counting c backward and b forward, that is $-x=+b-c$.

97. Laws of Addition and Subtraction. Starting with any symbol on the scale, we can now count forward or backward along the scale to any extent. We can therefore perform any series of additions and subtractions of positive numbers and the **Commutative Law** may be stated as follows:

In performing a series of operations which may be either additions or subtractions of positive numbers, *the result is not affected by the order of the operations.*

Here the restriction of Art 18, as to the operations being conducted in an order which is possible, is no longer necessary, for all orders are now possible.

The Associative Law. *If* a *is any number (positive or negative) and* b *and* c *are positive numbers, then*

$$a+(b+c)=a+b+c, \qquad\qquad (1)$$
$$a-(b+c)=a-b-c; \qquad\qquad (2)$$

and if b *is greater than* c,

$$a+(b-c)=a+b-c, \qquad\qquad (3)$$
$$a-(b-c)=a-b+c. \qquad\qquad (4)$$

We will prove (4) and leave the rest as an exercise.

$a-(b-c)$ denotes the last number counted in the following process: Starting with the number a on the scale, count $(b-c)$ numbers backward along the scale. At present, this operation is only *possible* when $(b-c)$ is a positive number, that is when b is greater than c, and this is the case by hypothesis.

Now, to count $(b-c)$ numbers backward is the same as counting b backward and c forward; hence

$$a-(b-c)=a-b+c.$$

98. Meaning of 3(−a). $3a$ is an abbreviation for $a+a+a$, and in the same way $3(-a)$ is an abbreviation for $-a-a-a$.

To find the number denoted by $-a-a-a$, we start with zero and count a numbers backward along the scale, then a more numbers backward, and then a more numbers backward; that is, we count altogether $3a$ numbers backward along the scale. The number finally reached is $-3a$; therefore

$$3(-a)=-a-a-a=-3a.$$

Thus "three times $(-a)$" is denoted by $3(-a)$ and has been shown to be the same number as $-3a$.

EXERCISE XXXV. MENTAL 1–37.

NOTE. *Letters, whose values are not given, stand for positive numbers.*

What is the value of

1. $5-8$. **2.** $-5-8$. **3.** $-8a-7a$.

4. $a-5a$. **5.** $a+7a-4a$. **6.** $a-2a+a$.

7. $a-5a+6$. **8.** $a-5a+6a$. **9.** $a-5+6$.

10. $-a+2a-3a$. **11.** $2a+3b-7a$. **12.** $a+5a-6a$.

13. $x-5x+6x$. **14.** $6(-x)-6x$. **15.** $5(-a)+3a$.

16. $2x-3x-10x+6x$. **17.** $-6x-8x-4x+20x$.

What are the following numbers in the scale?

18. The fifth number after 0.
19. The fifth number before 0.
20. The tenth number after 2.
21. The nth number after 2.
22. The tenth number before 2.
23. The nth number before 2.
24. The nth number before n.
25. The nth number after n.

What is the number

26. Greater than $x-3$ by 7?
27. Greater than $3-x$ by $5x$?
28. Less than $x-3$ by 7?
29. Less than $x-3$ by $5x$?

If $a=-5$ and $b=9$, state the values of

30. $a+b$.
31. $a-b$.
32. $a+2b$.
33. $a-2b$.

What value of x satisfies the following equations?

34. $-2x=6$.
35. $-x=2a-b$.
36. $3-x=2$.
37. $-x+6=-4$.

If $a=1$, $b=2$, $c=3$, $d=4$, find the value of

38. $a+2b-3c-4d$.
39. $a-2b-3c-4d$.
40. $-4a+3b-2c+d$.
41. $-4a-3b-2c-d$.

42. By performing the operations in the order indicated, verify the equality $a+(b-c)=a+b-c$, firstly when $a=10$, $b=8$, $c=7$, secondly when $a=-10$, $b=7$, $c=2$.

43. Prove the equality in the two cases of Ex. 42 by counting.

44. Prove the equality $a-(b-c)=a-b+c$ in the following cases: (i) when $a=10$, $b=8$, $c=7$, and (ii) when $a=-10$, $b=7$, $c=2$.

45. If $x=a+b$, the process of addition does not as yet enable us to find a value of x when $a=9$ and $b=-5$. Explain this.

46. Explain why our methods of subtraction do not enable us to say (as yet) that $5-(2-3)=5-2+3$.

47. Find the 5th and 10th terms of the Arithmetical progression
$$-x, (-x+2), (-x+4), \ldots.$$

48. Find the ath and bth term of the Arithmetical progression
$$(-a-3), (-a-1), (-a+1), \ldots.$$

49. If the 5th and 8th terms of an Arithmetical progression are $-a+2d$ and $-a+14d$ respectively, find the first term and the third term.

99. Use of Zero and Negative Numbers in Measurement. A gain of £a followed by a loss of £a results in no gain; in this case it is convenient to say "the gain is zero," and to write "Gain = £0."

Again, although we cannot speak of (-3) pennies as actually existing, the next example will show that *a gain of* (-3) *pence* may be reasonably taken to mean *a loss of 3 pence.*

Ex. 1. *Express as a gain the result of winning £3 and losing £5.*

"Gain" is found by subtracting what is lost from what is won.

$$\therefore \text{ gain in pounds} = 3 - 5 = (-2).$$

Thus, although £(-2), *standing by itself*, has no meaning whatever, yet a "*gain of* £(-2)" may be taken to mean a loss of £2.

Ex. 2. *Starting from a point O, I walk a miles eastwards to A, and then b miles westwards to B.* (1) *What is the distance from O to B measured eastwards?* (2) *Interpret the result when $a=3$ and $b=5$.*

(1) The distance from O to B measured eastwards is $(a-b)$ miles.

(2) When $a=3$ and $b=5$, $a-b=-2$, and in this case B is 2 miles *west* of O. If then *a distance of* (-2) *miles measured eastwards* is to have any meaning, it must mean *a distance of* $(+2)$ *miles measured westwards.*

We therefore make the following **convention** (or agreement) with regard to the meaning of the negative sign in the measurement of lengths:

If lengths set off in one direction or sense along a straight line are considered positive, then lengths set off in the opposite sense are to be considered negative.

Ex. 3. *Starting from a point O, I walk a miles eastwards and then b miles westwards, finally reaching a point B. Describe the position of B with regard to O* (1) *algebraically,* (2) *arithmetically.*

Algebraically. For all values of a and b, B is $(a-b)$ miles east of O; or the answer may be given thus: for all values of a and b B is $(b-a)$ miles west of O.

Arithmetically. If $a>b$, B is $(a-b)$ miles east of O, and if $a<b$, B is $(b-a)$ miles west of O.

The algebraical statement has therefore the advantage of *greater generality.*

EXERCISE XXXVI.

Express the results of the following (i) as a gain, (ii) as a loss :

1. Gain of £a followed by loss of £b.
2. Loss of £b followed by gain of £a.
3. Loss of £$7x$ followed by gain of £x.
4. Loss of £$10x$ followed by gain of £$(10x-2x)$.
5. What is the range of temperature for the year, if the maximum temperature was 92° (Fahrenheit), and the minimum 6° below zero (Fahrenheit)?
6. Reasonably interpret the following: (i) A is $(-x)$ miles north of B. (ii) The point A is (-2) metres above sea-level. (iii) A train is moving southwards at the rate of (-30) miles an hour. (iv) A owes B the sum of £(-10). (v) A has a handicap of $(-x)$ yards.
7. In a race of a yards, A's handicap was $(-x)$ yards and B's handicap was $(+y)$ yards. How far had A and B respectively to run before reaching the winning post?
8. The road from A to B rises a feet, falls b feet, falls c feet, rises d feet. (i) How high is B above A? (ii) How much is B below A?
9. At the end of each week I reckon in shillings money earned, and money spent, writing $+3$ for 3 shillings earned. At the end of a quarter, the results are as follows: 1st month, $+1-5-6+7$; 2nd month, $-3-2+8+12$; 3rd month, $-40-11+13+20$.

   Fill up the blanks in the table:

Month - -	1	2	3	For the quarter
Money saved				

10. The road from A to B (30 miles) rises a feet per mile for the first 10 miles, then falls b feet per mile for 5 miles, then falls c feet per mile for 4 miles and then rises d feet per mile for the rest of the way:

    (i) How high is A above B? (ii) How high is B above A?

    (iii) If $a=2$, $b=7$, $c=8$, $d=3$, how many feet is A below B?

    (iv) If $b=20$, $c=30$, $d=40$, and if A and B are both at the same level, what is the value of a? Give a reasonable interpretation to the result.

11. On the centigrade thermometer, the freezing point is 0° (0 degrees) and the boiling point of water is 100°: if "degrees of frost" means degrees below zero, express the results of the following in degrees of frost: (i) Temperature is 14° and falls 10°. (ii) Temperature is 12° and falls 20°. (iii) Temperature is -5° and falls 10°. (iv) Temperature is -5° and rises 20°.

12. The weight in water of a piece of iron, measured in ounces, is denoted by $+2$. The force necessary to keep a piece of cork under water is measured by a spring balance attached to the bottom of the vessel and found to be 2 ounces. How do you denote the weight in water of the cork?

13. If a stone is thrown vertically upwards with a velocity of u feet per second, and if after t seconds its velocity upwards is v feet per second, and the space, measured upwards, through which it has moved is s feet, it can be shown that

$$v=u-gt \quad \text{and} \quad s=ut-\tfrac{1}{2}gt^2,$$

where $g=32$.

Find v and s in the following cases, and interpret any results which may be negative or zero:

(i) $u=40,\quad t=2.$ (ii) $u=40,\quad t=3.$
(iii) $u=160,\quad t=4.$ (iv) $u=160,\quad t=5.$
(v) $u=192,\quad t=6.$ (vi) $u=160,\quad t=10.$

14. Observe that when a stone, thrown vertically upwards, reaches its highest point, it is for a moment at rest, that is to say, its velocity is zero. (i) Put $v=0$ in the formulae of Ex. 13, and find the corresponding values of t and s in terms of u. (ii) Show that a stone thrown upwards with velocity u continues to rise for $\left(\frac{u}{g}\right)$ seconds and that it rises through $\left(\frac{u^2}{2g}\right)$ feet.

15. A stone is thrown vertically upwards from a point O with a velocity of 80 feet per second; (i) After how many seconds is its velocity reduced to 16 feet per second? (ii) When is its velocity 16 feet per second downwards? What is the position of the stone at this moment? (iii) After how many seconds does the stone reach the point O again (that is to say, what is the time of flight)?

CHAPTER XII.

ADDITION AND SUBTRACTION.

100. Addition and Subtraction of Negative Numbers. It will be seen from the following examples that our definitions of addition and subtraction do not enable us to interpret the expressions $2+(-3)$ and $2-(-3)$:—

Ex. 1. *Find the value of* $(-3)+2$.

Here, 2 is to be added to -3; to do this, we start with the number -3 on the scale, and count two forward $(-2, -1)$, and we write
$$(-3)+2=-1.$$

Ex. 2. *Try to find the value of* $2+(-3)$.

(*First attempt.*) (-3) is to be added to 2. We are to start with 2 on the scale and count (-3) numbers forward. Now we cannot count (-3) numbers, so that $2+(-3)$ has no meaning at present.

(*Second attempt.*) If a gain of £2 is denoted by $+2$, then a loss of £3 is denoted by (-3). It *seems reasonable* to denote "a gain of £2 taken with a loss of £3 by $2+(-3)$. If we agree to do this, then $2+(-3)$ means the result of subtracting 3 from 2, and we conclude that $2+(-3)$ ought to mean $2-3$.

Now we have no right to make a property of abstract numbers depend on our notion about pounds and shillings. If $2+(-3)$ is the same as $2-3$, then this example is merely an illustration.

Ex. 3. *Try to find a meaning for* $2+0$.

Here we are directed to start with 2 and count 0 numbers forward. *If we take this to mean that we are to count no numbers*, then $2+0$ means the same as 2. If however there are *no* numbers to count, we cannot count them, and the assumption contained in the words in italics is unjustifiable.

101. Definition of Addition and Subtraction. The processes of Algebra will be much simplified if the letters used to denote number may stand for *any* numbers, whether positive, zero or negative. This will not be the case unless the letters used in the statements of the fundamental laws may stand for any numbers.

If we assume that the identity $a+b=b+a$ holds for all values of a and b, whether positive, zero or negative, and if we substitute 2 for a and (-3) for b, we shall have

$$2+(-3)=(-3)+2=2-3. \quad (\textit{Art. 100, Ex. 1})$$

Thus, to **add** (-3) is to **subtract** 3.

Again, *if we assume* that for all values of a and b, $a-b$ is **defined** by the equation

$$(a-b)+b=a,$$

then
$$\{2-(-3)\}+(-3)=2;$$
$$\therefore \{2-(-3)\}-3=2; \qquad (\textit{by the preceding})$$
$$\therefore 2-(-3)=2+3.$$

Hence, to **subtract** (-3) is to **add** 3.

Similarly it can be shown that

$$2+0=2=2-0.$$

The addition and subtraction of zero and of the negative number $(-x)$ (hitherto meaningless operations) are **defined** by

$$\mathbf{a+0 = a,}$$
$$\mathbf{a-0 = a,}$$
$$\mathbf{a+(-x) = a-x,}$$
$$\mathbf{a-(-a) = a+x,}$$

where a is any number whatever, positive, zero or negative.

It can be shown that with the above definitions, the Fundamental Laws of addition and subtraction hold good for all numbers, and that subtraction continues to be the inverse of addition.

The fundamental laws referred to are

(1) **The Commutative Law.** Additions and subtractions may be performed in any order.

(2) **The Associative Law.** This is contained in the equations

(i) $a+(b+c)=a+b+c$, (iii) $a+(b-c)=a+b-c$,

(ii) $a-(b+c)=a-b-c$, (iv) $a-(b-c)=a-b+c$.

Ex. 1. *Assuming the above definitions, prove that*

$$a-(b-c)=a-b+c$$

in the case when a, b, c are positive and $c>b$.

Let $x=c-b$, so that x is positive;

$$\therefore \quad -x=b-c; \qquad \textit{(Art. 96, Ex. 1)}$$

$$\begin{aligned} \therefore \quad a-(b-c) &= a-(-x) \\ &= a+x \qquad \textit{(Def.)} \\ &= a+(c-b) \\ &= a-b+c. \qquad \textit{(Art. 97)} \end{aligned}$$

102. Definitions. The **sum** of several numbers has been defined as the result of adding them together. This definition holds good for all numbers.

Thus, $2a+(-3b)+(-5)$ is the sum of the numbers $2a$, $-3b$ and -5, and means the same as $2a-3b-5$.

The number $(a-b)$ is called the **excess** of a over b; $(a-b)$ is also called the **algebraical difference** of a and b (or simply the **difference** of a and b). From Art. 95 it follows that a is greater than, equal to, or less than b according as $(a-b)$ is positive, zero or negative.

The **numerical difference** between two numbers a and b is denoted by $a \sim b$, and is the result of taking the less from the greater.

Thus the algebraical difference of -4 and 3 is $-4-3$.

The excess of -2 over 3 is $-2-3$ or (-5).

The expressions $5 \sim 7$, $7 \sim 5$, $7-5$ mean the same thing.

The numbers $(+2)$ and (-2) are said to have the same **absolute** or **numerical value**, namely 2.

When an expression is regarded as the *sum* of several numbers (positive, zero or negative), each of these numbers is called a **term** of the expression.

The expression $2a-3b-4c$, written as a *sum*, is

$$(+2a)+(-3b)+(-4c).$$

Hence the *terms* of the expression are $(+2a)$, $(-3b)$ and $(-4c)$.

The numbers 2, (-3), (-4) are called the **coefficients** of a, b, c in the expression $2a-3b-4c$.

Again, the expression $\frac{x+y}{4} - 3(x-y) - 4x \times y$ is the *sum* of $\frac{x+y}{4}$, $-3(x-y)$ and $-4x \times y$; these are therefore the terms of the expression.

Algebraical expressions are called **monomial, binomial, trinomial**, or **polynomial** expressions according as they contain *one*, *two*, *three* or *several* terms.

EXERCISE XXXVII. MENTAL 1–38.

What are the values of

1. $5+(-3)$. **2.** $5-(-3)$. **3.** $-5+(-3)$. **4.** $-5-(-3)$.

5. $5(-3)$. **6.** $-5(-3)$. **7.** $-3-(-5)$. **8.** $2-(4-6)$.

9. $(2-4)-6$. **10.** $-2-(4-6)$. **11.** $-2-4-6$. **12.** $-(-2-4)-6$.

If $a=-2$, what is the value of

13. $4a$. **14.** $4+a$. **15.** $4+2a$. **16.** $4-a$. **17.** $4-2a$.

Simplify the following :

18. $a+(3a-7a)$. **19.** $7a-(3a-7a)$. **20.** $3a-(3a-7a)$.

Subtract $(b-c)$ from each of the following :

21. Zero. **22.** $c-b$. **23.** $b+c$. **24.** $2a+2b$.

Subtract $(b+c)$ from each of the following :

25. $a+3b$. **26.** $a-3c$. **27.** Zero. **28.** $b-c$.

What is the excess

29. of $x+y$ over $x-y$? **30.** of $x-y$ over $x+y$?

31. of 5 over -2? **32.** of -2 over 5?

33. of $-3a$ over $4a$? **34.** of $-3a$ over $-4a$?

35. of a over -4? **36.** of a over 4?

37. of $2a$ over $(2a-b)$? **38.** of $2a$ over $(b-2a)$?

By substituting the value -4 for y, find the value of x when

39. $x+y=7$. **40.** $x-y=7$. **41.** $x+2y=7$. **42.** $x-2y=-7$.

43. $2x+y=8$. **44.** $4x-y=0$. **45.** $4x+y=0$. **46.** $4x-y=8$.

47. If $x+5y=11$ and $x-5y=-9$, what are the values of x and y?

48. If $x+z=8$ and $y+z=-4$, what is the value of $x-y$?

49. If $x-y=2$ and $y-z=3$, what is the value of $x-z$?

50. If $x-y=-5$ and $y+z=4$, what is the value of $x+z$?

103. Rules for Removing and Inserting Brackets. For practical purposes, the Associative Law is stated in the form of rules, thus :—

The value of an expression will be unaltered

(1) **If a bracket preceded by + is removed, each term that was within the bracket retaining its original sign.**

(2) **If a bracket preceded by − is removed, the sign of each term that was within the bracket being changed from + to −, or from − to +.**

Ex. 1. *Simplify* $4a-(a+2b-3c)+(3a+4b-2c-d)$.

The expression

$$=4a-a-2b+3c+3a+4b-2c-d \qquad (\textit{Associative Law})$$
$$=4a-a+3a-2b+4b+3c-2c-d \qquad (\textit{Commutative Law})$$
$$=6a+2b+c-d.$$

Again the value of an expression will be unaltered

(1) **If any set of terms of the expression is enclosed in a bracket preceded by +, each term retaining its original sign.**

(2) **If any set of terms is enclosed in a bracket preceded by −, the sign of each term enclosed in the bracket being changed from + to −, or from − to +.**

Ex. 2. *Find the sum of* $2a$, $3a$, $4a$, $-5a$, $-6a$, $-7a$.

The sum $=2a+3a+4a-5a-6a-7a$

$$=(2a+3a+4a)-(5a+6a+7a) \qquad (\textit{Associative Law})$$
$$=9a-18a$$
$$=-9a.$$

104. Method of Subtraction. The method of subtracting one expression from another will be understood from this example:

Ex. 1. *Subtract* $5a+2b-5c$ *from* $2a-3b-4c$.

The result $=(2a-3b-4c)-(5a+2b-5c)$

$$=2a-3b-4c-5a-2b+5c \qquad (\textit{Associative Law})$$
$$=2a-5a-3b-2b-4c+5c \qquad (\textit{Commutative Law})$$
$$=(2a-5a)-(3b+2b)+(-4c+5c) \qquad (\textit{Associative Law})$$
$$=-3a-5b+c.$$

When the dependence of each step on the Fundamental Laws is understood, the work may be arranged as follows:

$$(2a-3b-4c)-(5a+2b-5c)=2a-5a-3b-2b-4c+5c$$
$$=-3a-5b+c.$$

What is often called the **Rule of Subtraction** in Algebra will be understood from the following arrangement of the above:

To subtract $5a+2b-5c$ from $2a-3b-4c$, write the first expression under the second, thus:

$$\begin{array}{r} 2a-3b-4c \\ 5a+2b-5c \end{array}$$

Next, (mentally) change the signs of the lower line and add; the mental process is

$$-5a+2a=-3a; \quad -2b-3b=-5b; \quad +5c-4c=c,$$

and the work stands as follows:

$$\begin{array}{r} 2a-3b-4c \\ 5a+2b-5c \\ \hline -3a-5b+\ \ c \end{array}$$

105. The Vinculum. The expression $a-\overline{b+c-d}$ means the same as $a-(b+c-d)$. The line drawn over $b+c-d$ is called a **vinculum** and is frequently used instead of a bracket. Thus $2 \,.\, \overline{b-c}$ means the same as $2(b-c)$.

If an expression contains several brackets, in removing these we shall generally begin with the *innermost* bracket. The method will be understood from the following examples:

Ex. 1. *Simplify* (i) $5x-\{2(x+y)-3(y-x)\}$.

(ii) $a-2[b-3\{c-4(a-b)\}]$.

(i) The expression $=5x-\{2x+2y-3y+3x\}$

$$\begin{aligned} &=5x-\{5x-y\} \\ &=5x-5x+y \\ &=y. \end{aligned}$$

(ii) The expression $=a-2[b-3\{c-4a+4b\}]$

$$\begin{aligned} &=a-2[b-3c+12a-12b] \\ &=a-2[12a-11b-3c] \\ &=a-24a+22b+6c \\ &=-23a+22b+6c. \end{aligned}$$

EXERCISE XXXVIII.

Assuming the definitions given in Art. 101, verify the equalities:

(i) $a+(b+c)=a+b+c$, (ii) $a-(b+c)=a-b-c$,

(iii) $a+(b-c)=a+b-c$, (iv) $a-(b-c)=a-b+c$,

when a, b, c have the following values:

1. $a=3,\ b=2,\ c=-7$. **2.** $a=-3,\ b=2,\ c=-7$.

3. $a=3,\ b=-2,\ c=7$. **4.** $a=-3,\ b=-2,\ c=-7$.

5. $a=3,\ b=0,\ c=-5$. **6.** $a=-3,\ b=-5,\ c=0$.

Simplify the expressions (7–11), and refer to the law or definition which justifies each step:

7. $1-(2+3-8)$. **8.** $3a-5(-a)$ **9.** $5a-(2a+b-4c)-(5b+c)$.

10. Find the sum of $-a,\ 2a,\ -3a,\ -4a,\ 5a$.

11. Find the result of subtracting

(i) $a+2b-4c$ from $-a-5b+9c$. (ii) $-3b-6c$ from $a+b-3c$.

Work out the following examples in the shortest way you can:

12. Subtract $x+2y+3z$ from $5x-y-2z$.

13. Subtract $x-(2y-3z)$ from $(5x-y)+2z$.

14. Subtract $y-(2z+3x)$ from $z-(x+y)$.

15. Subtract $(a+b)-(c-d)$ from zero.

16. Subtract $(a-b)-(c-d)$ from $(d-c)-(b-a)$.

17. By how much does zero exceed $a-b-2c$?

18. By how much does $a-(b+c)$ exceed $(a-b)+c$?

19. What is the excess of $(a+b)-(c+d)$ over $(c-a)-(d-b)$?

20. Find the sum of $5x^2-3x-2a,\ 4x+3a-3x^2,\ a-2x^2-5x$.

21. Add together $a+(2b+3c),\ 2a-(b+2c),\ b-(c+a),\ c-(a+b)$.

22. Add together $4(x-2y+3z),\ -5(y-2z+3x),\ 6(z-2x+3y)$.

23. From $c(a-b)-x$ subtract $x-c(a+b)$.

24. What must be subtracted from $-2x+3y-5z$ in order that the remainder may be $2x-7y-4z$?

25. Add $5a-(7b-c)$ and $3b-(9a+c)$; subtract the sum from $c-4b$.

26. From $3x-(y-z)$ subtract $2z-(y-4x)$; add $2x+y$ to the result.

27. Add together

$$3x^2-2ax+5a^2,\ \ 3ax-7a^2+5x^2,\ \ 3a^2-7x^2-4ax;$$

and subtract $x^2-4ax+a^2$ from the sum.

28. Add together

$$6a^3-(5a^2b-b^3),\ \ 7a^2b-(3ab^2-2b^3),\ \ 5ab^2-(4a^3+b^3);$$

and subtract $2a^2b-2ab^2$ from the sum.

29. From $5x^3-6x^2+7x-8$ take the sum of

$$9x^3-7x^2-5x+3 \text{ and } -10x^3-8x^2+6x+4.$$

Simplify the following:

30. $4\{b-c-3(c-a)\}-5\{a-b-2(a-c)\}$.

31. $2\{a^2-(2ab+b^2)\}-4(b^2-ab-a^2)-\{6(a^2-b^2)-ab\}$.

32. $1-[2+\{3-(1+2)-3\}+1]-2$.

33. $3(a+c)-5(a+b+c)-[b-\{c+2a-(b-c)\}]$.

34. $2a-\{3a-(4b+2a)\}+\{5a-(4b-a)\}$.

35. $(x+y+z)-[4y+2z-\{3x-(3y-z)\}-2x]$.

36. $-[a-\{b-(c-x)-(b+x)\}+d]$ when a, b, c, d are all equal.

37. $2a-[c-(a-b+2c)]-[4b-\{3c+a-(\overline{4a+c}-5b)\}]$.

38. $2a-[3b-\{5c-(3a-5b)-(2c-5a)\}+2b]+3c$.

39. $a^2-[b^2-\{c^2-(2a^2-\overline{3b^2+a^2-c^2}-b^2)-c^2\}+c^2]$.

40. $5x-4[x+3\{2x-2(3x-\overline{1-x})\}]$.

41. $2x-3[y-\{3z+2x-(3y-z)+y\}-x]-z$.

106. Interpretation of the Negative Sign in Problems.

Ex. 1. *O, A, and B are points in the same vertical line, A is a feet above O, B is b feet above O, and C is the middle point of AB. Find the height of C above O and interpret the result when*

(1) $a=-2$, $b=6$, (2) when $a=-6$, $b=2$.

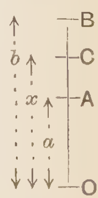

Fig. 18.

Let C be x feet above O, then since $AC=CB$,

$$\therefore\ x-a=b-x;\quad \therefore\ 2x=a+b,$$

$$x=\frac{a+b}{2}.$$

Observe that every step in the solution holds good whether the letters stand for positive or negative numbers, and that if

lengths measured upwards are denoted by positive numbers, then lengths measured downwards are denoted by negative numbers.

(1) If $a=-2$ and $b=6$, we have $x=\frac{-2+6}{2}=2$.

In this case, A is 2 feet below O, B is 6 feet above O, and C is 2 feet above O.

(2) If $a=-6$ and $b=2$, $x=\frac{-6+2}{2}=-2$.

Here A is 6 feet below O, B is 2 feet above O, and C is 2 feet below O.

Ex. 2. *At the present time, A's age is a years, B's age is b years, and in x years A will be twice as old as B. Find x in terms of a and b, and interpret the result when $a=30$ and $b=20$.*

In x years, A's age will be $(a+x)$ years and B's age will be $(b+x)$ years, hence

$$a+x=2(b+x),$$
$$x=a-2b.$$

Now every step in the solution holds good whether the letters stand for positive or negative numbers, and if $a=30$ and $b=20$,

$$x=30-2 \cdot 20=-10.$$

We interpret "(-10) years hence" as meaning "10 years ago" and thus obtain the solution of the following problem:— "A's age is now 30 years and B's age is 20 years; how many years ago was A's age double B's age?" It is seen that the answer is "10 years ago."

EXERCISE XXXIX.

1. OA and OB are lengths measured from a point O along a straight line, and C is the middle point of AB. If $OA=a$, $OB=b$ and $OC=x$, it has been shown that $2x=a+b$ (Art. 106, Ex. 1). Draw figures to illustrate this formula when (i) $a=-3$, $b=-7$; (ii) $a=3$, $b=-7$; (iii) $a=-3$, $b=7$.

2. The top of a pit-shaft is a feet above sea-level and the bottom of the shaft is b feet below sea-level; the point X is half way down the shaft, Y is two-thirds of the way down and Z is three-quarters of the way down. Find the depths of X, Y and Z below sea-level. If $a=900$ and $b=600$, how many feet are X, Y and Z respectively above or below sea-level?

3. OA, AB, BC are lengths set off along a straight line OX, and P is the middle point of OC. If $OA=a$, $AB=b$, $BC=c$ and $OP=x$, write down the equation connecting a, b, c and x.

4. In question 3, if a, b and c have the values given below, find the corresponding values of x, and in each case verify the result with a diagram: (i) $a=-2$, $b=4$, $c=6$. (ii) $a=-2$, $b=-4$, $c=6$. (iii) $a=-2$, $b=-4$, $c=10$. (iv) $a=-5$, $b=-3$, $c=4$.

5. At present A's age is a years and B's age is b years. How many years ago was A's age three times B's age? Interpret the result if $a=70$ and $b=20$.

6. A's age is x years and B's age is $4x$ years; $3a$ years hence A's age will be the same as B's age will be $3b$ years hence. Find x in terms of a and b.

7. Apply the formula obtained in Ex. 6 to answer the following: B's age is 4 times A's age and 30 years hence A's age will be the same as B's age was 6 years ago. What is A's present age?

8. A's age is x years and B's age is y years; a years hence A's age will be the same as B's was b years ago. Find y in terms of a, b and x. Interpret the result when $a=7$, $b=8$, $y=12$.

9. A and B play two rubbers. In the first rubber A wins a shillings and in the second B wins b shillings. In settling accounts A pays x shillings to B. What is the equation connecting a, b and x? Interpret the result when a is greater than b.

10. A is to pay twice as much as B towards the upkeep of a joint establishment. A and B respectively pay £a and £b towards the expenses; B lends £c to A and in settlement of accounts A pays £x to B. (1) Find x in terms of a, b and c. (2) If $a=600$, $b=150$, $c=50$, find the value of x and interpret the result.

11. A, B and C are to share equally the expenses of a certain shoot. A, B, C respectively pay £a, £b, £c towards the joint expenses. In settlement of accounts B pays £x to A and C pays £y to A. (1) Find x and y in terms of a, b and c. (2) Interpret the result when $a=90$, $b=300$, $c=450$.

12. A, B, C join in a shoot; B and C are to pay equal shares and A as much as B and C together. A, B, C respectively pay £a, £b, £c towards joint expenses. In settlement of accounts C pays £x to B and B pays £y to A. (1) Find x and y in terms of a, b, c. (2) Interpret the result when $a=100$, $b=200$, $c=300$.

CHAPTER XIII.

GRAPHICAL REPRESENTATION OF NUMBER.

(Continued.)

107. Representation of Numbers by Points. If an indefinite number of equal lengths are set off along a straight line

Fig. 19.

and the points of division are marked

$$\ldots \; -4, \; -3, \; -2, \; -1, \; 0, \; 1, \; 2, \; 3, \; 4, \ldots$$

in order, we obtain points which, by their position, represent the numbers of the complete scale of integers in these respects:

(1) for every number, there is one point and only one;

(2) the points occur in the order in which the corresponding numbers stand on the scale.

108. Representation of Pairs of Numbers by Points. Draw two straight lines $X'OX$, $Y'OY$ at right angles to one another. Choose some convenient unit of length, say one-tenth of an inch.

Starting from O, set off an indefinite number of lengths each equal to the unit of length along OX, OX', OY, OY'. Let the points of division along OX and OY be marked 1, 2, 3, 4, ... in order; let the points of division along OX', OY' be marked -1, -2, -3, -4, ... in order. (Fig. 20.)

To find the point which represents any pair of numbers (a, b), through the point a in $X'OX$ draw a parallel to $Y'OY$, and through the point b in $Y'OY$ draw a parallel to $X'OX$; these parallels meet in the point (a, b), which is taken to represent the pair of numbers (a, b).

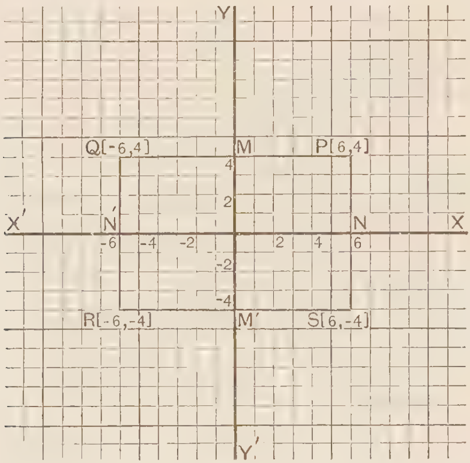

FIG. 20.

Thus in Fig. 20, P, Q, R, S are the points which represent the pairs of numbers $(6, 4)$, $(-6, 4)$, $(-6, -4)$, $(6, -4)$ respectively, and the points N, M, N', M' represent the pairs of numbers $(6, 0)$, $(0, 4)$, $(-6, 0)$, $(0, -4)$ respectively. The point O is called the **origin** and the lines $X'OX$, YOY' are called the **axes** of x and y respectively.

Points constructed in the manner just described represent pairs of numbers in the following respect:—

For every pair of numbers (either or both of which may be positive, negative or zero) there is one point and one only.

If P is any point in the plane of XOY, and if PN is drawn perpendicular to $X'OX$, the lengths ON, NP are called the **coordinates of P**; of these ON is called the **abscissa** and NP the **ordinate** of P. The abscissa ON is considered *positive or negative* according as it is measured along OX or along OX'.

If PM is drawn perpendicular to $Y'OY$, the ordinate NP is considered positive or negative according as M lies in OY or in OY'.

Thus it is seen that every point in the plane OXY, whose coordinates can be represented by numbers, corresponds to one pair of numbers and to one only.

In Fig. 20 the abscissa of R is -6 and the ordinate of R is -4.

Ex. 1. *Plot the points* $(-2, 5)$, $(-6, -3)$, $(8, -7)$ *and find the area of the triangle of which these are the vertices.*

In Fig. 21 $\triangle PQR =$ trapezium $MNPQ + \triangle NRP - \triangle MRQ$.

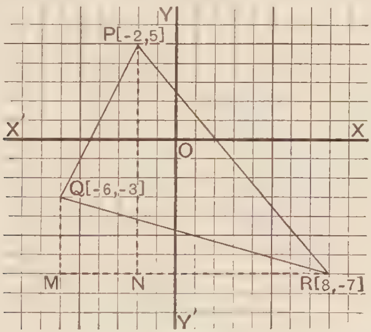

FIG. 21.

$\therefore$ area of $\triangle PQR$

$$= \tfrac{1}{2}(MQ + NP) \,.\, MN + \tfrac{1}{2} NR \,.\, NP - \tfrac{1}{2} MR \,.\, MQ$$
$$= \tfrac{1}{2}(4 + 12) \,.\, 4 + \tfrac{1}{2} \,.\, 10 \,.\, 12 - \tfrac{1}{2} \,.\, 14 \,.\, 4$$
$$= 64 \text{ units of area.}$$

Ex. 2. *If* ξ, η *are the coordinates of the middle point* (R) *of the straight line joining the points* (P, Q), *whose coordinates are* (x, y), (x', y'), *then* $\xi = \frac{1}{2}(x + x')$, $\eta = \frac{1}{2}(y + y')$.

Construct as in Fig. 22; show that $\triangle$s QKR, RHP are congruent, and hence that $QK = RH$ and $KR = HP$.

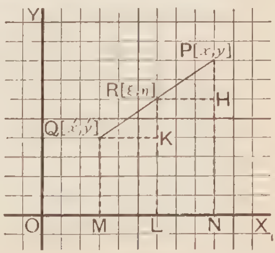

FIG. 22.

Now $\quad QK = \xi - x', \quad RH = x - \xi,$

$$\therefore \quad \xi - x' = x - \xi,$$
$$\therefore \quad 2\xi = x + x',$$
$$\therefore \quad \xi = \tfrac{1}{2}(x + x').$$

Similarly $\qquad \eta = \frac{1}{2}(y + y')$.

Ex. 3. *In the equation $x+2y=6$, give to x the values -10, -4, 0, 6, 10, and find the corresponding values of y. Plot the points which represent these pairs of values of x and y. Verify that the points lie in a straight line. Take any other point on the line whose coordinates can be represented by numbers, and verify that these numbers satisfy the equation $x+2y=6$.*

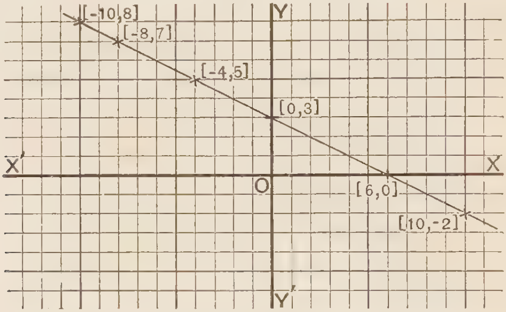

Fig. 23.

Corresponding values of x and y are shown in the table:

x	-10	-4	0	6	10
y	8	5	3	0	-2

Plotting the points $(-10, 8)$, $(-4, 5)$, ..., we find by using a ruler that the points lie in a straight line. It is seen that the point $(-8, 7)$ is on the line, and when $x=-8$ and $y=7$, we have $x+2y=(-8)+2 \cdot 7=6$.

$\therefore$ the values $(-8, 7)$ of x and y satisfy $x+2y=6$.

109. The Linear Equation (*continued*). Experiments in plotting points, whose coordinates satisfy a linear equation, point to the fact that the conclusions of Art. 87 are also true when x and y have any integral values, whether positive, zero, or negative.

To the facts stated in Art. 87 may be added the following:

1. The points where a straight line cuts the axes of x and y respectively are to be found by putting $y=0$ and $x=0$ in the equation to the line.

2. The ordinate of every point in $X'OX$ is zero; the **equation to the axis of x** is therefore $y=0$; similarly, **the equation to the axis of y** is $x=0$.

EXERCISE XL.

1. Draw the triangles whose vertices are the points given in (i)-(iv). Calculate the areas of the triangles.

(i) $(0, 0)$, $(0, 7)$, $(-8, 0)$; (ii) $(0, -4)$, $(-5, 4)$, $(-8, -2)$;
(iii) $(6, -3)$, $(-1, 3)$, $(-4, -9)$; (iv) $(-2, 6)$, $(-5, -8)$, $(7, -4)$.

2. Draw the quadrilaterals whose vertices are the points given in (i)-(iv). Calculate the areas of the quadrilaterals.

(i) $(0, 3)$, $(-2, -4)$, $(6, -4)$, $(8, 3)$;
(ii) $(0, 0)$, $(-3, -5)$, $(0, -12)$, $(10, 5)$;
(iii) $(6, -10)$, $(6, -3)$, $(-4, 3)$, $(-4, 0)$;
(iv) $(-2, -10)$, $(10, -7)$, $(-4, 6)$, $(-6, 1)$.

3. Prove that the distance of the point (x, y) from the origin is $\sqrt{x^2+y^2}$. Illustrate geometrically the inequality $x+y>\sqrt{x^2+y^2}$, where x, y stand for positive numbers.

4. By substitution in the formula

"Distance of (x, y) from the origin $=\sqrt{x^2+y^2}$,"

find the distance of each of the following points from the origin and verify by drawing diagrams.

(i) $(-6, -8)$; (ii) $(12, -5)$; (iii) $(-15, 8)$; (iv) $(-20, -21)$.

5. Prove that the distance between the points (x, y), (a, b) is

$$\sqrt{(x-a)^2+(y-b)^2}.$$

6. By substitution in the formula of Ex. 5, find the distance between the points given in (i)-(iii). Verify by drawing diagrams.

(i) $(-10, -3)$, $(14, -10)$; (ii) $(-5, 7)$, $(7, -9)$;
(iii) $(12, 18)$, $(0, -17)$.

7. By using the formulae of Art. 108, Ex. 2, find the coordinates of the middle points of the straight lines joining the points in (i)-(iv). Illustrate by diagrams.

(i) $(3, 2)$, $(7, 8)$; (ii) $(2, 2)$, $(8, -10)$;
(iii) $(3, 3)$, $(-7, -3)$; (iv) $(3, 2)$, $(-5, -10)$.

8. Find the coordinates of the points where the straight lines, whose equations are given in (i)-(iv), cut the axes; plot these points and draw the lines.

(i) $5x+6y=90$; (ii) $3x-4y=36$;
(iii) $3y-2x=24$; (iv) $4x+7y=-84$.

9. In equations (i)–(ix), (*a*) Give to x in succession the values -8, -6, 0, 6; find the corresponding values of y and plot the points which represent these pairs of values.

(*b*) Verify that the four points of each set lie on a straight line.

(*c*) Read off from the diagram the abscissa of the point on each line whose ordinate is -15. Verify that the coordinates of the point satisfy the equation to the line.

(i) $x+y=0$; (ii) $3x+2y=0$; (iii) $x+y=19$;
(iv) $y-x=3$; (v) $y-2x=3$; (vi) $2y-x-2=0$;
(vii) $x+2y=32$; (viii) $2y-3x=6$; (ix) $3x+2y=42$.

Solve graphically the following simultaneous equations:

10. $3x-4y=-1$
$4x+y=24$.

11. $5x+6y=2$
$3x+y=-17$.

12. $5x+7y=0$
$x+y=-8$.

13. $5(x+8)+4(y+6)=0$
$2x+3y+34=0$.

14. Prove that the straight lines whose equations are $3x-4y=6$, $x-3y=12$, $2y-5x=24$ pass through the same point. Verify by a diagram.

15. If the lines whose equations are $3x+5y=2$, $x+y=-2$, $x+ay=10$ pass through the same point, find the value of a. Verify by a diagram.

16. If (x, y) is a point which is equidistant from the points (a, b), (a', b'), prove that (x, y) satisfy the equation

$$2x(a-a')+2y(b-b')=a^2-a'^2+b^2-b'^2.$$

17. What is the equation to the perpendicular bisector of the line joining the points (a, b), (a', b')?

CHAPTER XIV.

MULTIPLICATION, DIVISION, POWERS AND ROOTS.

110. Zero as a Multiplier. In accordance with the definition of multiplication in Art. 5,

$$0 \times 3 = 0 + 0 + 0 = 0;$$

but the definition attaches no meaning to the expression 3×0. If, however, the identity $ab = ba$ is to hold when one of the letters stands for zero, and if we substitute 3 for a, and 0 for b, then

$$3 \times 0 = 0 \times 3 = 0.$$

Assuming the Commutative Law for Multiplication to hold when one of the factors is zero, we are led to the following definition :

DEF. $$\mathbf{a \times 0 = 0 \times a = 0,}$$

or *the result of multiplying any number by zero is zero.*

Hence $a \,.\, 0 \,.\, b = (a \,.\, 0) \,.\, b = 0 \,.\, b = 0,$

and $a \,.\, b \,.\, 0 = (ab) \,.\, 0 \;\; = 0;$

thus $a \,.\, b \,.\, 0 = a \,.\, 0 \,.\, b \;\; = 0 \,.\, a \,.\, b = 0:$

and generally :

If any factor of an expression is zero, then the expression is zero.

It is easy to see that, with the above definition, the Distributive, Commutative and Associative Laws of Multiplication hold when any of the numbers concerned is zero.

111. Zero in Division. If $cb=a$, then, by the definition of division, $c=\frac{a}{b}$: if this definition is to hold, when either b or c stands for zero, then

$$\text{(i) since } 0\times a=0\,;\ \therefore\ 0=\frac{0}{a}\,;$$

$$\text{(ii) since } a\times 0=0\,;\ \therefore\ a=\frac{0}{0}.$$

Thus, *the expression* $\frac{0}{0}$ *may stand for* **any number**, and its value is said to be indeterminate.

Again, if $b\times 0=a$, then a is zero, and we conclude that there is no number which, if multiplied by zero, will give as the product any number but zero; hence *unless a is zero, the expression* $\frac{a}{0}$ *is meaningless, and can represent* **no number whatever.**

Thus, at present, **we are unable to assign any definite meaning to the operation of dividing a number by zero.**

112. A Negative Multiplier. In accordance with the definition of multiplication in Art. 5,

$$(-4)\times 3=-4-4-4=-(4\times 3),$$

but the definition attaches no meaning to either of the expressions $3\times(-4)$ or $(-3)\times(-4)$.

Now the identity

$$\mathbf{(a-b)(x-y)=ax-ay-bx+by}$$

has been proved to hold when a, b, x, y stand for positive numbers, and $a>b$ and $x>y$. If the letters used in Algebra to denote numbers may stand for *any* numbers, whether positive, zero or negative, then this identity must hold for *all* values of a, b, x, y. *Assuming this to be the case*, we have:

(1) If $b=0$ and $y=0$, then

$$(+a)(+x)=+(ax)-a\times 0-0\times x+0\times 0;$$

$$\therefore\ \mathbf{(+a)(+x)=+ax.}$$

(2) If $a=0$ and $y=0$,

$$(-b)(+x)=0\times x-0\times 0-(bx)+b\times 0;$$

$$\therefore \mathbf{(-b)(+x)=-(bx).}$$

(3) If $b=0$ and $x=0$,

$$(+a)(-y)=a\times 0-(ay)-0\times 0+0\times y;$$

$$\therefore \mathbf{(+a)(-y)=-(ay).}$$

(4) If $a=0$ and $x=0$,

$$(-b)(-y)=0\times 0-0\times y-b\times 0+(by);$$

$$\therefore \mathbf{(-b)(-y)=+(by).}$$

The results (1) and (2) are in accordance with the definition of multiplication as explained above, and the results (3), (4) are to be regarded as *defining* the operation of multiplying by a negative number.

The results (1)–(4) are generally stated as in the following article.

113. Rule of Signs. **The product of two factors with like signs is positive, and the product of two factors with unlike signs is negative; thus**

$$\mathbf{(+a)(+b)=+(ab);\quad (-a)(-b)=+(ab),}$$

$$\mathbf{(+a)(-b)=-(ab);\quad (-a)(+b)=-(ab).}$$

It can be shown that, with the definitions of multiplication just given, the Distributive, Commutative and Associative Laws for multiplication hold good for all numbers, whether positive or negative.

Particular cases of this statement are proved below, and the student should have no difficulty in supplying the proof of any other case. The justice of these definitions will be *confirmed* by the fact that, as the student advances, he will find that they never lead to inconsistent results.

Ex. 1. *Prove that $(ab)c=a(bc)$ in the case when a, b, c are all negative.*

Let $a=-x$, $b=-y$, $c=-z$, then x, y, z are all positive, and

$$\begin{aligned}(ab)c&=\{(-x)(-y)\}(-z)\\&=\{xy\}(-z) &&(\textit{Rule of Signs})\\&=-xyz. &&(\textit{Rule of Signs})\end{aligned}$$

Again, $a(bc) = (-x)\{(-y)(-z)\}$

$= (-x)\{yz\}$ (*Rule of Signs*)

$= -x(yz)$ (*Rule of Signs*)

$= -xyz$; (*Associative Law for Positive Numbers*)

$\therefore\ (ab)c = a(bc).$

Ex. 2. *Assuming the definitions contained in the Rule of Signs for Multiplication by a Negative Number, prove that*

$$a(b-c) = ab - ac$$

in the case when a *is negative,* b *and* c *are positive and* $b < c$.

Let $a = -x$ and $b - c = -y$; then, by hypothesis, x and y are positive; also $x = -a$ and $y = c - b$; hence

$a(b-c) = (-x)(-y)$

$= +(xy)$ (*Rule of Signs*)

$= x(c-b)$

$= xc - xb$ (*Distributive Law for Positive Numbers*)

$= (-a)c - (-a)b$

$= -ac - \{-(ab)\}$ (*Rule of Signs*)

$= ab - ac.$

114. Powers of a Negative Number. If x stands for any number, $(-x)^2 = (-x)(-x)$

$= x^2,$ (*Rule of Signs*)

$(-x)^3 = (-x)(-x)(-x)$

$= (x^2)(-x)$ (*Rule of Signs*)

$= -x^3.$ (*Rule of Signs*)

Continuing the process, it is seen that $(-x)^4 = x^4$, $(-x)^5 = -x^5$; and, in general,

if n is **even** $(-x)^n = \boldsymbol{+}x^n$,

if n is **odd** $(-x)^n = \boldsymbol{-}x^n$.

The three Index Laws contained in the identities

$$\mathbf{a^m \cdot a^n = a^{m+n};\quad (a^m)^n = a^{mn};\quad (ab)^m = a^m b^m}$$

depend on the Commutative and Associative Laws of Multiplication, and therefore hold whether a, b stand for positive or negative numbers.

Ex. 1. *Find the value of $3a^2b^3c^4$ when $a=-1$, $b=-2$, $c=-5$.*

Here $$3a^2b^3c^4=3(-1)^2(-2)^3(-5)^4.$$

We first find the sign of the result *by counting the total number of negative factors*: (-1) occurs twice, (-2) occurs three times and (-5) occurs four times. The total number of negative factors is therefore $2+3+4$ or 9; and since 9 is an *odd* number, by the Rule of Signs, the sign of the result is *minus*. Hence

$$\begin{aligned} 3a^2b^3c^4 &= -3.1^2.2^3.5^4 \\ &= -3.(2.5)^3.5 \\ &= -3.1000.5 \\ &= -15000. \end{aligned}$$

EXERCISE XLI.

Simplify

1. $(-x)(-y)(-z)$. **2.** $(-3x)^2$. **3.** $(-2x)^3$.

4. $(-2)^3(-x)^3$. **5.** $(-x)(-x^2)$. **6.** $(-x)^2(-x^2)$.

7. $(-1)^5$. **8.** $(-1)^6$. **9.** $(-3)^2-(-2)^3$.

10. $1-(-1)^2$. **11.** $2+(-2)^2$. **12.** $\{2+(-2)\}^2$.

13. If n is any positive integer, state the values of

(i) $(-1)^{2n}$; (ii) $(-1)^{2n+1}$; (iii) $(-1)^{2n+2}$.

14. If $a=-1$, $b=-2$, $c=3$, $d=-5$, find the values of

(i) $2a^2b^2$. (ii) $a^5b^3c^2$. (iii) $-5a^4b^2c^2$.

(iv) $-8b^2c^2d^3$. (v) $25a^4b^4c^4$. (vi) $bc+ca+ab$.

(vii) $a^2+b^2+c^2$. (viii) $a^3+b^3+c^3-3abc$.

(ix) $(b+c)(c+a)(a+b)+abc$. (x) $a^2+b^2+c^2+2bc+2ca+2ab$.

15. By substituting 1 for a, -1 for b, and -2 for c, verify the following identities :—

(i) $a^3-b^3=(a-b)(a^2+ab+b^2)$.

(ii) $(bc+ca+ab)(a+b+c)=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.

(iii) $(b+c)(c+a)(a+b)=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$

(iv) $(b-c)(c-a)(a-b)=-[a^2(b-c)+b^2(c-a)+c^2(a-b)]$.

16. Remembering that the square of any number is positive, whether the number itself is positive or negative, prove that for all values of x and y which are unequal

$$x^2+y^2>2xy.$$

17. Prove the identity $(x+y)^2=(x-y)^2+4xy$. Hence show that if x and y are two numbers whose sum is given, their product is

greatest when they are equal. Illustrate this by writing down the product of every pair of numbers whose sum is 10.

18. Prove that if x and y are two numbers whose product is given, their sum diminishes as their difference diminishes. Illustrate this by writing down the sum and difference of every pair of factors of 24.

19. Prove the identity

$$(y-z)^2+(z-x)^2+(x-y)^2=2(x^2+y^2+z^2-yz-zx-xy).$$

Hence show that for all values of x, y, z which are not all equal

$$x^2+y^2+z^2>yz+zx+xy.$$

Illustrate by putting $x=1$, $y=2$, $z=3$.

20. Assuming the definitions contained in the "Rule of Signs," prove that $a(b+c)=ab+ac$ in the case when a and b are negative, c is positive and $(b+c)$ is negative.

115. A Negative Divisor. If division, when the divisor is a negative number, is *defined* as the *inverse of multiplication*, it follows that

$$\text{(i)}\ \frac{-\mathbf{a}}{\mathbf{b}}=-\frac{\mathbf{a}}{\mathbf{b}};\quad \text{(ii)}\ \frac{\mathbf{a}}{-\mathbf{b}}=-\frac{\mathbf{a}}{\mathbf{b}};\quad \text{(iii)}\ \frac{-\mathbf{a}}{-\mathbf{b}}=+\frac{\mathbf{a}}{\mathbf{b}}.$$

Proof of (i). Let $\frac{a}{b}=c$;

$\therefore\ cb=a.$ (*Def. of Division*)

Also $(-c)b=-(cb)$ (*Rule of Signs*)

$\therefore\ (-c)b=-a.$

Hence, if division is to be the inverse of multiplication,

$$\frac{-a}{b}=-c=-\frac{a}{b}.$$

The proofs of (ii) and (iii) are similar to that of (i).

The Rule of Signs is therefore often stated as follows:—

In multiplication and division like signs produce plus and unlike signs produce minus.

The Laws of Division, as stated in Chapter VIII., and the Theorems proved in that chapter depend on the Definition of Division and on the Fundamental Laws of Multiplication; these Laws and Theorems therefore hold good for all numbers whether positive or negative.

Ex. 1. *Simplify*

$$\text{(i)}\ \frac{y-x}{x-y};\quad \text{(ii)}\ \frac{a^2-b^2}{ab-a^2};\quad \text{(iii)}\ \frac{10x+3y-\dfrac{18y^2}{x}}{6-\dfrac{5x}{y}}.$$

(i)
$$\frac{y-x}{x-y}=\frac{-(x-y)}{x-y}=-\frac{x-y}{x-y}=-1.$$

(ii)
$$\frac{a^2-b^2}{ab-a^2}=\frac{a^2-b^2}{a(b-a)}$$
$$=\frac{(a+b)(a-b)}{-a(a-b)}$$
$$=-\frac{(a+b)(a-b)}{a(a-b)}$$
$$=-\frac{a+b}{a}.$$

(iii)
$$\frac{10x+3y-\dfrac{18y^2}{x}}{6-\dfrac{5x}{y}}=\frac{\left(10x+3y-\dfrac{18y^2}{x}\right)xy}{\left(6-\dfrac{5x}{y}\right)xy}$$
$$=\frac{(10x^2+3xy-18y^2)y}{(6y-5x)x}$$
$$=-\frac{(5x-6y)(2x+3y)y}{(5x-6y)x}$$
$$=-\frac{(2x+3y)y}{x}.$$

Ex. 2. *Find the value of* $\dfrac{3b^3c^3d^5}{2a^3}$, *when* $a=-2,\ b=4,\ c=-5,\ d=-1.$

Here
$$\frac{3b^3c^3d^5}{2a^3}=\frac{3.4^3(-5)^3.(-1)^5}{2(-2)^3}.$$

We first find the *sign* of the result by *counting the total number of negative factors in the dividend and the divisor*: (-5) occurs 3 times, (-1) occurs 5 times, (-2) occurs 3 times; thus the total number of negative factors is $3+5+3$ or 11. The number 11 is *odd*; hence, by the Rule of Signs, the sign of the result is $-$, and

$$\frac{3b^3c^3d^5}{2a^3}=-\frac{3.4^3.5^3.1^3}{2.2^3}=-\frac{3.2^6.5^3}{2^4}=-3.2^2.5^3=-1500.$$

EXERCISE XLII.

NOTE. *Letters, of which the values are not given, are supposed to have such values that the operations indicated are possible.*

Simplify the following :

1. $\dfrac{ay-ax}{bx-by}$. 2. $\dfrac{a^2-a}{1-a}$. 3. $\dfrac{(b-a)^2}{a-b}$. 4. $\dfrac{x^2-4}{(2-x)^2}$.

5. $\dfrac{(xy^2-x^2y)^2}{x^2y^2(x-y)}$. 6. $(a+b)\div\left(1+\dfrac{b}{a}\right)$. 7. $(a-b)\div\left(1-\dfrac{a}{b}\right)$.

8. $(a+b)\div\left(1-\dfrac{a^2}{b^2}\right)$. 9. $(a^3-b^3)\div\left(1-\dfrac{a}{b}\right)$. 10. $\dfrac{a}{a-b}+\dfrac{b}{a-b}$.

11. $\dfrac{a}{a-b}-\dfrac{2b}{b-a}$. 12. $\dfrac{b^2-a^2}{a+b}-\dfrac{b^2-a^2}{a-b}$. 13. $\dfrac{b^3-a^3}{a-b}+\dfrac{b^3+a^3}{a+b}$.

14. $\dfrac{\dfrac{a}{b}-\dfrac{c}{d}}{\dfrac{b}{d}-\dfrac{a}{c}}$. 15. $\dfrac{\dfrac{b}{c}-a}{\dfrac{ac}{b}-1}$. 16. $\dfrac{x-\dfrac{16}{x}}{\dfrac{4}{x}+1}$. 17. $\dfrac{x-7+\dfrac{12}{x}}{\dfrac{3}{x}-1}$.

18. $\dfrac{2x^2-5xy+2y^2}{\dfrac{2y}{x}-1}$. 19. $\dfrac{6x-17y+\dfrac{12y^2}{x}}{\dfrac{4y}{x}-3}$. 20. $\dfrac{6x^2-xy-12y^2}{3-\dfrac{2x}{y}}$.

If $a=-2$, $b=-1$, $c=-5$, $d=-4$, find the values of

21. $\dfrac{2a-d}{2a}$. 22. $\dfrac{c^3d^2}{a^4b^5}$. 23. $\dfrac{(abcd)^2}{d^3}$. 24. $\dfrac{a^3+b^3}{a+b}$. 25. $\dfrac{a^6-b^6}{a^3-b^3}$.

26. $\dfrac{c^2-(3b)^2-d^2}{a^2d^2}$. 27. $\dfrac{a^4+a^2b^2+b^4}{ab-(a^2+b^2)}$. 28. $\dfrac{a^5+1}{a+1}$. 29. $\dfrac{a^2-(b-c)^2}{a-b+c}$.

116. Square Roots. Since $(-x)^2=x^2$, we see that

(i) A square root of a number being defined as a number whose square is the given number, it follows that both $+x$ and $-x$ are square roots of x^2. It will be convenient to call $+x$ **the** square root of x^2.

(ii) The square of any number, whether positive or negative, is a positive number, and there is no number (of a class yet considered) whose square is negative. Instead, however, of saying that a negative number has no square root, we invent a new class of numbers called **complex** or **imaginary** numbers, and we

say that the square root of a negative number is a number belonging to this new class. Numbers which are not imaginary are called **real**.

Established custom is the only justification for restricting the word *imaginary* to refer to *complex* numbers; correctly speaking, *all* numbers are imaginary.

117. Cube Roots. It has been shown that $(-x)^3 = -x^3$; moreover, there is no number, positive or negative, other than $(-x)$, whose cube is $(-x^3)$: **the** cube root of $(-x^3)$ will therefore be defined as $(-x)$, and we shall write $\boldsymbol{\sqrt[3]{-x^3} = -x}$.

It can be shown that every number, positive or negative, has *three* cube roots, two of which are imaginary numbers.

118. Illustrations. (i) The number 16 has two square roots, namely $+4$ and -4; **the** square root of 16 is $+4$, and we write

$$\sqrt{16} = +4.$$

(ii) There is no number, of a class yet considered, whose square is (-16); such an expression as $\sqrt{-16}$ denotes an imaginary number.

(iii) Since $(-3)^3 = -27$ and $(-2)^5 = -32$, we have

$$\sqrt[3]{-27} = -3 \text{ and } \sqrt[5]{-32} = -2.$$

Ex. 1. *Find the value of* $\sqrt[3]{-4 \,.\, 14^2 \,.\, 28}$.

$$4 \,.\, 14^2 \,.\, 28 = 4 \,.\, 2^2 \,.\, 7^2 \,.\, 4 \,.\, 7 = 2^6 \,.\, 7^3;$$

$$\therefore \sqrt[3]{-4 \,.\, 14^2 \,.\, 28} = -\sqrt[3]{2^6 \,.\, 7^3} = -2^2 \,.\, 7 = -28.$$

Ex. 2. *Find two values of x which satisfy the equation*

$$(2x+7)^2 = 25.$$

Here $2x+7$ must stand for a number whose square is 25; such numbers are 5 and -5. Hence we may have

$$2x+7=5 \quad \text{or} \quad 2x+7=-5;$$

$$\therefore x=-1 \text{ or } x=-6;$$

and the required values of x are -1 and -6.

119. Roots of Algebraical Expressions. If A stands for a given algebraical expression, it *may happen* that A is a perfect nth power, that is to say an expression B *may* exist such that $B^n = A$. In this case, B is called an nth root of A.

Observing that

$$(a-b)^2=(b-a)^2=a^2-2ab+b^2,$$

we see that both $(a-b)$ and $(b-a)$ are square roots of $a^2-2ab+b^2$.

In general, *an expression which is a perfect square has two square roots, which differ only in sign.* It will be convenient to call $(a-b)$ **the** square root of $a^2-2ab+b^2$.

Ex. 1. *Find the cube root of $125(a+b)^6 \div 8x^3y^{12}$.*

$$\sqrt[3]{\frac{125(a+b)^6}{8x^3y^{12}}}=\sqrt[3]{\frac{5^3(a+b)^6}{2^3x^3y^{12}}}=\frac{5(a+b)^2}{2xy^4}. \qquad \textit{(See Art. 77)}$$

Ex. 2. *Find the square root of $(4x^2-1)(2x^2-x-1)(2x^2-3x+1)$.*

The expression $=(2x+1)(2x-1)(2x+1)(x-1)(2x-1)(x-1)$

$$=(2x+1)^2(2x-1)^2(x-1)^2;$$

$\therefore$ Square root $=(2x+1)(2x-1)(x-1)$.

Ex. 3. *Find the square root of*

(i) $25x^2-70xy+49y^2$; (ii) $9x^2+4y^2+25z^2-20yz-30zx+12xy$.

Here we use the identities

$$(a+b)^2=a^2+2ab+b^2; \quad (a-b)^2=a^2-2ab+b^2.$$

(i) $25x^2-70xy+49y^2=(5x)^2-2(5x)(7y)+(7y)^2=(5x-7y)^2$;

$\therefore$ Square root $=5x-7y$.

(ii) Arrange the expression in descending powers of x, thus:

The expression $=9x^2+6x(2y-5z)+(4y^2-20yz+25z^2)$

$$=(3x)^2+2(3x)(2y-5z)+(2y-5z)^2$$

$$=(3x+2y-5z)^2;$$

$\therefore$ Square root $=3x+2y-5z$.

EXERCISE XLIII.

Simplify

1. $\sqrt[3]{-8a^3}$.
2. $\sqrt[4]{(-9a^6)(-9a^2)}$.
3. $\sqrt[5]{-32a^{15}}$.
4. $\sqrt[2n]{(-a)^{2n+1}(-a)^{2n-1}}$.
5. $\sqrt[2n+1]{-a^2(-a)^{4n}}$.
6. $\sqrt[3]{-21952}$.
7. $\sqrt[5]{-2.3.16.27^3}$.
8. $\sqrt[3]{-9.24.28.98}$.
9. $\sqrt[3]{-49.75.315}$.
10. $\sqrt{\dfrac{x^6y^8}{a^{10}}}$.
11. $\sqrt[3]{\dfrac{x^6y^9}{a^{15}}}$.
12. $\sqrt{\dfrac{x^{2n}y^{2n+2}}{(a+b)^2}}$.
13. $\sqrt[3]{(x-1)^2(1-x)}$.
14. What are the two square roots of $a^2+2ab+b^2$?

Find the square roots of

15. $4x^2-12x+9$. **16.** $25x^2+80xy+64y^2$. **17.** $16x^4-56x^2+49$.

18. $4x^6-36x^3yz+81y^2z^2$. **19.** $(x^2-3x+2)(x^2-4x+3)(x^2-5x+6)$.

20. $(3x^2+2x-1)(3x^2-10x+3)(x^2-2x-3)$.

21. $(5x^2+x-6)(10x^2-3x-18)(2x^2-5x+3)$.

22. $4x^2+y^2+25z^2-10yz+20zx-4xy$.

23. $9x^2+4y^2+16z^2+16yz+24zx+12xy$.

24. $x^2-4xy+4y^2+2x-4y+1$. **25.** $9x^4+30x^2y^2+25y^4-12x^2-20y^2+4$.

26. $x^2y^2+6x^2y+9y^2+4xy+12x+4$.

Find two values of x which satisfy the following equations:

27. $x^2=a^2$. **28.** $(x+a)^2=b^2$. **29.** $(2x-5)^2=25$. **30.** $(2x-5)^2=49$.

31. $(2x+5)^2=(x+1)^2$. **32.** $x^4=16$. **33.** $x^6=729$. **34.** $x^{2n}=1$.

Find a value of x which satisfies the following equations:

35. $x^3=-8$. **36.** $(x+a)^3+27b^3=0$. **37.** $x^5+a^5=0$. **38.** $x^{2n+1}+1=0$.

120. Rules for Equalities. The Rules for Equalities stated in Art. 16 are

(1) If $\mathbf{a=b}$, then $\mathbf{b=a}$. (2) If $\mathbf{a=b}$ and $\mathbf{b=c}$, then $\mathbf{a=c}$.

If $\mathbf{a=b}$, then

(3) $\mathbf{a+x=b+x}$; (5) $\mathbf{a\times x=b\times x}$;

(4) $\mathbf{a-x=b-x}$; (6) $\mathbf{a\div x=b\div x}$.

If $\mathbf{a=b}$ and $\mathbf{x=y}$, then

(7) $\mathbf{a+x=b+y}$; (9) $\mathbf{a\times x=b\times y}$;

(8) $\mathbf{a-x=b-y}$; (10) $\mathbf{a\div x=b\div y}$.

These Theorems and their Converses are true if the letters stand for any positive or negative numbers.

Any of the letters may also stand for zero, *so long as zero is not used as a divisor*; thus (6) and (10) become meaningless if x and y are zero. Again, the converse of (5) is as follows: "If $ax=bx$, then $a=b$." This does not hold if x is zero. Thus we have $3\times 0=0=4\times 0$, but it does not follow from this that $3=4$.

121. Rules for Inequalities. In Rules 1–4, 7, 8, 11 and 13 in Art. 22, the letters may have zero or negative as well as positive values.

Rules 5 and 6 must be stated as follows: *If a and b stand for any numbers such that $a>b$, then if x is positive, $ax>bx$ and $a\div x>b\div x$; if x is negative, $ax<bx$ and $a\div x<b\div x$.*

122. Permanence of Algebraic Form. The truth of every algebraical identity depends on (i) the Rules for Equalities, (ii) the laws which govern the operations of addition, subtraction, multiplication and division. These rules and laws are the same for the class of numbers consisting of zero and negative numbers as for the class consisting of natural numbers, excepting only that zero is not to be used as a divisor. We conclude that *in the identities proved in Part I., the letters may stand for any numbers, positive, zero or negative, so long as zero does not occur as a divisor.*

We may also state the matter thus: *If a number of algebraical operations are performed, the* **form** *of the result is the same for all values of the letters concerned, provided that zero is not used as a divisor.*

Ex. 1. *From the identity*

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad \text{..................}(1)$$

deduce the identity

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3. \quad \text{..................}(2)$$

The identity (1) is true for all values of a and b, whether positive or negative; we may therefore substitute $(-b)$ for b; thus

$$\{a+(-b)\}^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3;$$

$$\therefore\ (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Ex. 2. *From the identity*

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2yz + 2zx + 2xy$$

deduce the expanded form of $(2a-5b-3c)^2$.

In the given identity, the letters may stand for any numbers, whether positive, zero or negative; we may therefore substitute $(2a)$ for x, $(-5b)$ for y and $(-3c)$ for z; thus

$$\begin{aligned}\{(2a)+(-5b)+(-3c)\}^2 &= (2a)^2 + (-5b)^2 + (-3c)^2 + 2(-5b)(-3c) \\ &\quad + 2(-3c)(2a) + 2(2a)(-5b);\end{aligned}$$

$$\begin{aligned}\therefore\ (2a-5b-3c)^2 &= 4a^2 + 25b^2 + 9c^2 + 2(15bc) \\ &\quad + 2(-6ca) + 2(-10ab) \\ &= 4a^2 + 25b^2 + 9c^2 + 30bc - 12ca - 20ab.\end{aligned}$$

Ex. 3. *Write down the expansion of* $(a-2b-3c+d)^2$.

As in Art. 37, we have

$$\begin{aligned}(a-2b-3c+d)^2 &= \{a+(-2b)+(-3c)+d\}^2 \\ &= a^2+(-2b)^2+(-3c)^2+d^2+2a(-2b) \\ &\quad +2a(-3c)+2ad+2(-2b)(-3c) \\ &\quad +2(-2b)d+2(-3c)d \\ &= a^2+4b^2+9c^2+d^2-4ab-6ac+2ad \\ &\quad +12bc-4bd-6cd.\end{aligned}$$

Ex. 4. *From the identity* $\dfrac{a^2-b^2}{a-b}=a+b$ *deduce the value of* $\dfrac{(y-3z)^2-(x-2y)^2}{3(y-z)-x}$, *and mention any exceptional case.*

In the given identity, the letters may stand for any numbers, positive, zero or negative, *so long as a is not equal to b* (for in this case zero would be used as a divisor); we may therefore substitute $(y-3z)$ for a and $(x-2y)$ for b, so that

$$\begin{aligned}a^2-b^2 &= (y-3z)^2-(x-2y)^2, \\ a-b &= (y-3z)-(x-2y) \\ &= 3(y-z)-x, \\ a+b &= (y-3z)+(x-2y) \\ &= x-y-3z;\end{aligned}$$

$\therefore$ the given expression $=x-y-3z$, unless $3(y-z)-x=0$, in which case the expression is meaningless.

EXERCISE XLIV.

1. Write down the expansions of

(i) $(a-b+c)^2$; (ii) $(a-b-c)^2$;

(iii) $(2a-b-3c)^2$; (iv) $(-3a-4b+5c)^2$;

(v) $(a-b+c-d)^2$; (vi) $(a-b-c-d)^2$;

(vii) $(a-2b-5c+d)^2$; (viii) $(a-2b-3c+4d)^2$.

2. Expand the following, arranging the results in descending powers of x:

(i) $(x^2-2x-1)^2$; (ii) $(x^3-3x-4)^2$;
(iii) $(x^3-2x^2-3x+4)^2$; (iv) $(x^3-x^2-5x-3)^2$;
(v) $(x^2+ax-b)^2$; (vi) $(x^3-ax^2-b)^2$.

3. Simplify the expressions:

(i) $(b-c)^2+(c-a)^2+(a-b)^2+2(c-a)(a-b)+2(a-b)(b-c)$
$+2(b-c)(c-a)$;

(ii) $(x+y+z)^2+(y+z-x)^2+(z+x-y)^2+(x+y-z)^2$.

4. By making proper substitutions in the identity $\dfrac{a^2-b^2}{a+b}=a-b$, simplify

(i) $\dfrac{(2x-y)^2-(x-2y)^2}{3(x-y)}$, (ii) $\dfrac{(y+z-x)^2-(z+x-y)^2}{2z}$,

(iii) $\dfrac{(a+b-c-d)^2-(a-b+c-d)^2}{2(a-d)}$.

5. You are given the identity $a^3+b^3=(a+b)(a^2-ab+b^2)$. Explain how to derive the identity

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

6. Simplify $\dfrac{(x-y)^3+1}{(x-y)^2-x+y+1}$.

7. Prove that if $b-c=0$, or if $c-a=0$, or if $a-b=0$, then each of the following expressions is zero:

(i) $a^2(b-c)+b^2(c-a)+c^2(a-b)$;
(ii) $a^3(b-c)+b^3(c-a)+c^2(a-b)$.

8. Prove that the expression

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$$

is zero (i) if $b+c=0$; (ii) if $c+a=0$; (iii) if $a+b=0$.

9. If $x+y+z=0$, prove that $x^3+y^3+z^3-3xyz=0$.

10. By means of the result in Ex. 9, prove the following:

(i) If $x=2y-3z$, then $x^3-8y^3+27z^3+18xyz=0$.
(ii) $(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y)$.
(iii) $x^3(y-z)^3+y^3(z-x)^3+z^3(x-y)^3=3xyz(y-z)(z-x)(x-y)$.
(iv) $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$
$=3(b+c-2a)(c+a-2b)(a+b-2c)$.

11. Prove that the expression

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$

is zero (i) if $a+b+c=0$, (ii) if any one of the numbers a, b, c is equal to the sum of the other two.

CHAPTER XV.

MISCELLANEOUS FACTORS. H.C.F. AND L.C.M.

NOTE. *At this point the student is recommended to revise Chapter VI.*

123. Miscellaneous Examples on Factors.

Ex. 1. *Factorize the following expressions:*

(i) $46xy - 15x^2 - 35y^2$; (ii) $x^2 - x(a - 6b) - 5b(a - b)$.

(i) $$46xy - 15x^2 - 35y^2 = -(15x^2 - 46xy + 35y^2)$$
$$= -(3x - 5y)(5x - 7y).$$

(ii) Employing the method of Art. 41, we seek two expressions whose product is $5b(a - b)$ and whose difference is $a - 6b$; these are $a - b$ and $5b$, and

$$x^2 - x(a - 6b) - 5b(a - b) = x^2 - x(a - b - 5b) - 5b(a - b)$$
$$= \{x^2 - x(a - b)\} + \{5bx - 5b(a - b)\}$$
$$= x\{x - (a - b)\} + 5b\{x - (a - b)\}$$
$$= (x - a + b)(x + 5b).$$

Ex. 2. *Factorize the following expressions:*

(i) $9x^4 - 25x^2y^2 + 16y^4$; (ii) $9x^4 - 49x^2y^2 + 16y^4$.

(i) $$9x^4 - 25x^2y^2 + 16y^4 = (x^2 - y^2)(9x^2 - 16y^2)$$
$$= (x + y)(x - y)(3x + 4y)(3x - 4y).$$

(ii) It will be found on trial that the method just employed does not enable us to factorize $9x^4 - 49x^2y^2 + 16y^4$; we therefore try the method of Art. 45, Ex. 4, and endeavour to express $9x^4 - 49x^2y^2 + 16y^4$ as the difference of two squares. Now

$$(3x^2 - 4y^2)^2 = 9x^4 - 24x^2y^2 + 16y^4;$$
$$\therefore\ 9x^4 - 49x^2y^2 + 16y^4 = (3x^2 - 4y^2)^2 - 25x^2y^2$$
$$= (3x^2 + 5xy - 4y^2)(3x^2 - 5xy - 4y^2).$$

Ex. 3. *Show that $(a-b)$ is a factor of $a^2(b-c)+b^2(c-a)$, and prove that*

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b).$$

$$\begin{aligned} a^2(b-c)+b^2(c-a) &= (a^2b-ab^2)-(ca^2-cb^2) \\ &= ab(a-b)-c(a^2-b^2) \\ &= (a-b)\{ab-c(a+b)\}; \end{aligned}$$

$$\begin{aligned} \therefore\ a^2(b-c)+b^2(c-a)+c^2(a-b) & \\ &= (a-b)\{ab-c(a+b)\}+c^2(a-b) \\ &= (a-b)\{c^2-c(a+b)+ab\} \\ &= (a-b)(c-a)(c-b). \end{aligned}$$

Now $(c-b)=-(b-c)$;

$$\therefore\ a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b).$$

EXERCISE XLV.

A.

Express the following as the product of as many factors as possible :

1. $9x^2-18x$. **2.** ax^2+bx. **3.** $pam^2x^3y^4-qn^3x^4y^3$.

4. $a(b-3)-b+3$. **5.** $ac+bc+ad+bd$. **6.** $ac+bd-ad-bc$.

7. $a^2-ab+ac-bc$. **8.** $2bd+ac+2ad+bc$. **9.** $2px+3qy+qx+6py$.

10. $12x^2+3xy-8zx-2yz$. **11.** $xy+4x-9y-36$.

12. $p^2+p(m-n)-mn$. **13.** $x^2-2a-x(2-a)$.

14. $x^3+x^2y+xy^2+y^3$. **15.** $6x^3+4x^2y+9xy^2+6y^3$.

16. $6(x^3+y^3)-xy(9x+4y)$. **17.** $(a^2-b^2)+(a+b)$.

18. $a^2-3a-4b^2-6b$. **19.** $p(x^2-1)-x(p^2-1)$.

20. $x(x-1)-y(y-1)$. **21.** $a(a+c)-b(b+c)$.

22. $(a+c)(b-d)-(a+d)(b-c)$. **23.** $ax+bx+cx+ay+by+cy$.

24. $(ax-by+az)-(bx-ay+bz)$. **25.** $(x+1)(x+2)^2+(x+2)(x+1)^2$.

26. $a(a-b+c-d)-b(c-d)$. **27.** $a(a^2-c^2)+b(c-a)(b-c)$

28. Show that $(a+b+c)$ is a factor of each of the following expressions, and in each case find the other factor :

(i) $a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)$;

(ii) $bc(b+c)+ca(c+a)+ab(a+b)+3abc$.

29. Show that $(b-c)$ is a factor of $b^3(c-a)+c^3(a-b)$, and prove that

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$$

30. Prove that $(b+c)$ is a factor of $b^2(c+a)+c^2(a+b)+2abc$, and hence show that
$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc=(b+c)(c+a)(a+b).$$

31. By substituting x^2 for a, y^2 for b and z^2 for c in the identity proved in Art. 123, Ex. 3, factorize the expression :
$$x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2).$$

B.

Express as the product of as many factors as possible :

1. $10xy-x^2-25y^2$.
2. $26xy-(x^2+169y^2)$.
3. x^4-9x^2-70.
4. $x^2-2x-323$.
5. x^6-20x^3-96.
6. $x^2y^2-4xy-77$.
7. $a^4b^4+a^3b^3-20a^2b^2$.
8. $a^2x^2-a^3x-30a^4$.
9. $x^2+10x-119$.
10. $x^2-x-182$.
11. $x^2-xyz-240y^2z^2$.
12. $x^2-3x-340$.
13. $5x^2-10x-315$.
14. $44xy-242y^2-2x^2$.
15. $90x^2-33xyz+3y^2z^2$.
16. $x^4-28x^2yz+187y^2z^2$.
17. $x^2+(a-b)x-ab$.
18. $x^2-2x(a+2b)+8ab$.
19. $x^2+(a+b+c)x+c(a+b)$.
20. $x^2-(a+5)x+3(a+2)$.
21. $x^2-(a+b-c)x-c(a+b)$.
22. $x^2-(a+1)x-2(a+3)$.
23. $1-a+b-110(a-b)^2$.
24. $1-3a-3b-4(a+b)^2$.

C.

Express as the product of as many factors as possible :

1. $39x-18-20x^2$.
2. $6x^2-17xyz-45y^2z^2$.
3. $15x^2-16xy-15y^2$.
4. $20x^2-x-30$.
5. $24x^3-2x^2-15x$.
6. $18x^2+53x-35$.
7. $36x^2-42x+12$.
8. $3x^2+31x+56$.
9. $54x^2-51xy-14y^2$.
10. $24x^2+41x-35$.
11. $24x^2+70xy-75y^2$.
12. $4x^4+20x^2+25$.
13. $9x^2+55xy-136y^2$.
14. $25x^4y-40x^3y^2+16x^2y^3$.
15. $108x^2-24x-175$.
16. $15(x-y)(x+y)-16xy$.
17. $18x^2-84xy+98y^2$.
18. $90xy-25x^2-81y^2$.
19. $60xy-9x^2-100y^2$.
20. $2(a+b)^2-7(a+b)(x+y)+3(x+y)^2$.
21. $-9(a-b)^2+30(a-b)c-25c^2$.
22. $prx^2+(ps+qr)x+qs$.
23. $px^2+(pq+1)x+q$.
24. $px^2+(pq-1)x-q$.
25. $(x^2-1)(x+2)+(x^2+2x)(x+1)$.
26. $x(x+1)(x-1)-6x-6$.

D.

Express as the product of as many factors as possible :

1. $(8a-6b)^2-(6b-4c)^2$.
2. $(a-2b+3c)^2-(a-2b-3c)^2$.
3. $(3a+b-2c)^2-(a-2b+3c)^2$.
4. $(a+b+c)^2-4(b-c)^2$.
5. $y^2+4z^2-a^2-x^2-4yz+2ax$.
6. $x(x^2-4)+5(x^2-4)$.
7. $x^2(x+3)-4(x+3)$.
8. $a^2-2ab+2bc-c^2$.
9. $(a^2-b^2)-(a-b)^2$.
10. $a^2x^2-b^2y^2-b^2x^2+a^2y^2$.
11. $a^2x^2+b^2y^2-b^2x^2-a^2y^2$.
12. $a^2-b^2+2bc-c^2$.
13. $a^3+a^2b-ab^2-b^3$.
14. $1-3x-3xy-y^2$.
15. x^3-x^2-x+1.
16. $x^2+ax+2ay-4y^2$.
17. x^3+x^2-4x-4.
18. x^4-y^4.
19. $(x+y)^4-(x-y)^4$.
20. $(x+y)^4-(x^2-y^2)^2$.
21. $(bx-ay)^2-(ax-by)^2$.
22. $(2x-yz)^2-(zx-2y)^2$.
23. x^4-5x^2-36.
24. x^4-18x^2+81.
25. x^4-13x^2+36.
26. x^4-25x^2+144.
27. $(x+2)(x^2-1)+(x+1)(x^2+2x+3)$.
28. $(x^2-1)(x+2)+(x^2-4)(x-1)$.
29. $4x^4+81$.
30. x^4-18x^2+49.
31. x^4-98x^2+1.
32. $x^4-102x^2y^2+y^4$.
33. $4x^4+8x^2y^2+9y^4$.
34. $4x^4-21x^2y^2+25y^4$.
35. Prove the identity

$$4a^2b^2-(a^2+b^2-c^2)^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

Hence express $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$ as the product of four factors.

E.

Express as the product of as many factors as possible :

1. $(a+2b)^3-(a-2b)^3$.
2. $(5a+3b)^3+(3a+5b)^3$.
3. $(2x+y)^3-(3x-2y)^3$.
4. $a^3x-b^3y+a^3y-b^3x$.
5. $a^3x+b^3y+a^3y+b^3x$.
6. $x^5+x^3y^2-x^2y^3-y^5$.
7. x^4-x^3+x-1.
8. $x^5-x^3-x^2+1$.
9. $x^2(x+1)-y^2(y+1)$.
10. $x^3+y^3+3xy(x+y)$.
11. $x^3+5x^2y+5xy^2+y^3$.
12. $x^3-y^3+(x+y)(x^2-2xy+y^2)$.

124. Highest Common Factor and Lowest Common Multiple. If it is possible to find all the factors of two or more algebraical expressions, the H.C.F. and L.C.M. of the expressions can be at once found by comparing the results as in Arts. 31, 32.

Ex. 1 *Find the* H.C.F. *and* L.C.M. *of* x^2+4x+3 *and* x^2-4x-5.

$$x^2+4x+3=(x+1)(x+3),$$
$$x^2-4x-5=(x+1)(x-5);$$

$$\therefore \text{ H.C.F.}=x+1; \quad \text{L.C.M.}=(x+1)(x+3)(x-5).^*$$

Ex. 2. *Find the* H.C.F. *and* L.C.M. *of*

$$12x^3y(x^2-y^2)^2, \quad 15(x^2y-xy^2)^3, \quad 6(x^3y^2+x^2y^3)^2.$$

$$12x^3y(x^2-y^2)^2 = 12x^3y\{(x+y)(x-y)\}^2 = 12x^3y(x+y)^2(x-y)^2,$$
$$15(x^2y-xy^2)^3 = 15\{xy(x-y)\}^3 = 15x^3y^3(x-y)^3,$$
$$6(x^3y^2+x^2y^3)^2 = 6\{x^2y^2(x+y)\}^2 = 6x^4y^4(x+y)^2;$$

$$\therefore \text{ H.C.F.}=3x^3y; \quad \text{L.C.M.}=60x^4y^4(x+y)^2(x-y)^3.$$

Ex. 3. *Find the* H.C.F. *and* L.C.M. *of* $a^4+2a^3+a^2$, a^3-a^2-a+1, a^4-2a^3+2a-1.

$$a^4+2a^3+a^2 = a^2(a^2+2a+1) = a^2(a+1)^2,$$

$$\begin{aligned} a^3-a^2-a+1 &= (a^3-a^2)-(a-1)\\ &= a^2(a-1)-(a-1)\\ &= (a^2-1)(a-1)\\ &= (a+1)(a-1)(a-1)\\ &= (a+1)(a-1)^2.\end{aligned}$$

$$\begin{aligned} a^4-2a^3+2a-1 &= a^4-1-(2a^3-2a)\\ &= (a^2+1)(a^2-1)-2a(a^2-1)\\ &= (a^2+1-2a)(a^2-1)\\ &= (a-1)^2(a-1)(a+1)\\ &= (a-1)^3(a+1);\end{aligned}$$

$$\therefore \text{ H.C.F.} = a+1; \qquad \text{L.C.M.} = a^2(a+1)^2(a-1)^3.$$

---

* NOTE. In practical applications of the above method, it will generally be found that the result is required to be expressed as the product of its simplest factors.

125. Theorem. The product of any two algebraical expressions is equal to the product of the H.C.F. and L.C.M. of the given expressions.

Proof. Let X and Y denote any two algebraical expressions, and let H and L be respectively the H.C.F. and L.C.M. of X and Y; then, by hypothesis, H is the factor of highest degree which is common to both X and Y; hence, if

$$X = Hx \quad \text{and} \quad Y = Hy,$$

it follows that x and y have no common factor other than unity;

$$\therefore \; L = Hxy;$$

$$\therefore \; HL = H^2xy = (Hx)(Hy) = XY.$$

It follows from this theorem that *if H is known, L can be found by dividing one of the given expressions by H and multiplying the result by the other expression.*

EXERCISE XLVI.

1. Find the factor common to $3x^2 - 4x + 1$ and $4x^4 - 5x^3 + x^2$.

Find the H.C.F. of

2. $x^2 - 3x + 2$ and $x^3 - x - 6$.
3. $10x^2 - x - 21$ and $14x^2 - 11x - 15$.
4. $3x^2 + x - 2$ and $3x^2 + 4x - 4$.
5. $2x^2 + 7x - 15$ and $x^2 + 9x + 20$.
6. Find the simplest expression which contains both $x^2 + 2x - 3$ and $x^2 - 3x + 2$ as factors.

In the following examples, find the L.C.M. of the given expressions; find also the H.C.F. in the cases where this exists.

7. $ax + ay$ and $ax - ay$.
8. $x^2 - 1$ and $x^2 - 3x + 2$.
9. $4a(a + b)$ and $6ab(a^2 - ab - 2b^2)$.
10. $a^3 - a^2b$, $a^2b + ab^2$, $a^2b - b^3$.
11. $ab^3c^4(x - y)^2$, $b^4c^4z(x - y)$, $b^5(x - y)^3z^2$.
12. $a^2 - ab$, $b^2 + ab$, $a^3 - b^3$, $a^2 - b^2$, $a^3 + b^3$.
13. $4a^3b^2(x + y)$, $12a^4b(x^2 - y^2)$, $10a^2b^3(x + y)^3$.
14. $30a^2b^2c(a^3 - b^3)$, $36a^2bc^3(a - b)^2$, $48a^3b^2c(a^2 - b^2)$.
15. $2(x^2 - y^2)$, $3(x^3 - y^3)$, $4(x^4 - y^4)$.
16. $6xy^2(x + y)$, $3x^2(x - y)^3$, $4y(x^4 - y^4)$.
17. $27(x - 3)^2$, $24(x + 4)^2$, $3(x^2 + x - 12)$.
18. $3x(x^2 - 2x - 35)$, $2x(x^2 - 4x - 45)$.
19. $4x^3y - 12x^2y^2 + 9xy^3$, $4x^4 - 9x^2y^2$, $4x^2y^2 - 6xy^3$.

20. $2x^2-x-6,\ 3x^2-7x+2,\ 6x^2+7x-3.$

21. $x^2+ax-2a^2,\ x^2-2ax+a^2,\ 2x^2-ax-a^2.$

22. $2x^2-x-6,\ 2x^2+5x+3,\ x^3-x^2-2x.$

23. $8a^3-18ab^2,\ 8a^3+8a^2b-6ab^2,\ 8a^2-2ab-b^2.$

24. $6x^4-11x^3+3x^2,\ 3x^3-7x^2+2x,\ 2x^2-7x+6.$

25. $a^4b-4b^5,\ a^5+3a^3b^2+2ab^4,\ a^3b+2ab^3,\ a^4-a^2b^2-2b^4.$

26. $2x^2-12x+10,\ 2x^3-11x^2+5x,\ 2x^4-3x^3+x^2.$

27. $10x^2+13x-3,\ 15x^2-13x+2,\ 5x^2+14x-3.$

28. $5xy-6(x^2-y^2),\ 5xy+6(x^2-y^2),\ 13xy-6(x^2+y^2).$

29. $x^2-1,\ x^2-6x-7,\ x^3-3x^2+2x,\ 3x^2-7x+2.$

30. $x^2-a^2+2ab-b^2,\ x^2+2ax+a^2-b^2.$

31. $2abc,\ b(a-c),\ 4a(ac+ab-bc-c^2).$

32. $2x^2-x-3,\ x^2-2x-3,\ x^3-3x^2-x+3.$

33. $x^2+x-2,\ x^2-3x+2,\ x^3+x^2-4x-4.$

34. $x^2-(y+z)^2,\ y^2-(z+x)^2,\ z^2-(x+y)^2.$

35. $a^2+2ab+b^2-c^2$ and $a^2+2ac-b^2+c^2.$

36. $4x^3-x^2-4x+1,\ 3x^3-3x^2+x-1.$

37. $x^3-ax^2+a^2x-a^3,\ x^3+ax^2+a^2x+a^3,\ x^3+ax^2-a^2x-a^3.$

38. $(2a^2+3y^2)x+(2x^2+3a^2)y,\ (2a^2-3y^2)x+(2x^2-3a^2)y.$

39. $4x^3-8x^2+3x-6$ and $12x^3+4x^2+9x+3.$

40. $x^4-y^4,\ x^3+y^3,\ x^4+x^2y^2+y^4.$

41. $x^2-1,\ x^3-1,\ x^3+1,\ x^4+x^2+1.$

42. $8x^3+27y^3,\ 8x^3-27y^3,\ 16x^4+36x^2y^2+81y^4.$

43. $x^4+x,\ x^4-x^2,\ x^5-x^2,\ x^5+x^3+x.$

44. $a^4+a^2b^2+b^4,\ a^3-b^3,\ a^3+b^3,\ (a^2-b^2)^2.$

45. Find the H.C.F. of

$$14x^2-5(a-b)x-(a-b)^2 \text{ and } 21x^2-(11a+3b)x-2a(a-b).$$

46. Find the H.C.F. of $(x-2)(2x^2-7x+6)+(x-2)(2x-3)$
and $(x-1)(2x^2-7x+6)-(x-1)(2x-3).$

47. Find the L.C.M. of $x^2+(2b-a)x-2ab,\ x^2-(2b+a)x+2ab,$
$x^2+(2b+a)x+2ab,\ x^2-(2b-a)x-2ab.$

CHAPTER XVI.

LONG MULTIPLICATION AND DIVISION.
THE DIVISION TRANSFORMATION.

126. Long Multiplication. By the Distributive Law

$$(2x+3)(4x-5)=(2x+3)(4x)+(2x+3)(-5);$$

so that to multiply $(2x+3)$ by $(4x-5)$, we multiply $(2x+3)$ by $(4x)$ and by (-5) in succession, and add the results. The work may be arranged as follows:

$$\begin{array}{l} 2x + 3 \\ 4x - 5 \\ \hline 8x^2 + 12x \\ \qquad -10x - 15 \\ \hline 8x^2 + 2x - 15 \end{array}$$

Ex. *Multiply* $5a+2a^3-3a^2-7$ *by* $-1+3a^2+2a$.

We arrange *both expressions in descending powers of* a, *or both in ascending powers of* a and proceed as on the left hand side below:

$$\begin{array}{lllllll|lllllll} 2a^3 & -3a^2 & +5a & -7 & & & & 2 & -3 & +5 & -7 & & & \\ 3a^2 & +2a & -1 & & & & & 3 & +2 & -1 & & & & \\ \hline 6a^5 & -9a^4 & +15a^3 & -21a^2 & & & & 6 & -9 & +15 & -21 & & & \\ & 4a^4 & -6a^3 & +10a^2 & -14a & & & & 4 & -6 & +10 & -14 & & \\ & & -2a^3 & +3a^2 & -5a & +7 & & & & -2 & +3 & -5 & +7 & \\ \hline 6a^5 & -5a^4 & +7a^3 & -8a^2 & -19a & +7 & & 6 & -5 & +7 & -8 & -19 & +7 & \end{array}$$

$$\therefore \text{ product} = 6a^5 - 5a^4 + 7a^3 - 8a^2 - 19a + 7.$$

We can shorten the work by *detaching the coefficients*, keeping each in its proper relative place, as on the right hand side above:

If we employ this method of "Detached Coefficients," when we supply the letters in the last step, it must be noticed that the degree of the product is the sum of the degrees of the given expressions, that is $3+2$ or 5; thus the first term in the product is $6a^5$.

127. Homogeneous Expressions.

Theorem. If A and B are homogeneous expressions of the mth and nth degrees respectively in x, y, z ..., then the product of A and B is a homogeneous expression of degree m + n in x, y, z, ...

For by the definition of a homogeneous expression (Art. 29), *every* term in A is of degree m and every term in B is of degree n. Also *any* term in the product AB is obtained by multiplying some term in A by some term in B.

Hence the degree of every term in AB is $m+n$, that is AB is homogeneous and of degree $m+n$.

Expressions which are homogeneous in two letters can be conveniently multiplied together by the method of *detached coefficients.*

Ex. *Multiply* $x^3+2xy^2-3y^3$ *by* $x^3-x^2y+2y^3$.

The expressions are arranged in descending powers of x, and if we proceed by the method of detached coefficients, it must be noted that the first expression has no term containing x^2 and the second has no term containing x. These terms must be supplied with zero coefficients, and we multiply

$$x^3+0x^2y+2xy^2-3y^3 \text{ by } x^3-x^2y+0xy^2+2y^3.$$

$$\begin{array}{l}
1+0+2-3 \\
\underline{1-1+0+2} \\
1+0+2-3 \\
\quad -1-0-2+3 \\
\qquad\qquad +2+0+4-6 \\
\hline
1-1+2-3+3+4-6
\end{array}$$

Since each of the given expressions is homogeneous and of the third degree in x and y, the product is homogeneous and of the sixth degree in x and y. The coefficients, standing in the above order, therefore belong to x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6;

$$\therefore \text{ product} = x^6-x^5y+2x^4y^2-3x^3y^3+3x^2y^4+4xy^5-6y^6.$$

128. Product of Expressions containing several Letters. In multiplying together expressions which contain several letters, the work can frequently be shortened *by choosing one letter* and arranging both expressions in ascending powers, or both in descending powers *of this letter.*

Ex. *Find the product of* $a^2+b^2+c^2-bc-ca-ab$ *and* $a+b+c$.

Long Method.

$$
\begin{array}{llllllll}
a^2 + & b^2 + & c^2 - & bc - ca - ab & & & & \\
a + & b + & c & & & & & \\
\hline
a^3 + ab^2 + ac^2 - & & abc - ca^2 - a^2b & & & & & \\
\quad -ab^2 & & -\ abc & & + a^2b + b^3 + bc^2 - b^2c^* & & & \\
 & -ac^2 - & abc + ca^2 & & & -bc^2 + b^2c + c^3 & & \\
\hline
a^3 & & -3abc & & +b^3 & +c^3 & &
\end{array}
$$

* In multiplying by a, the terms are written down in the order in which they are obtained; in multiplying by b and c, the terms are arranged so that *like* terms occur in vertical columns.

Short Method. Arrange the multiplicand and the multiplier in descending powers of a; thus:

$$
\begin{array}{llll}
a^2 - a(b+c) + (b^2 - bc + c^2) & & & \\
a + (b+c) & & & \\
\hline
a^3 - a^2(b+c) + a(b^2 - bc + c^2) & & & \\
\qquad a^2(b+c) - a(b^2 + 2bc + c^2) + (b^3 + c^3) & & & \\
\hline
a^3 \qquad -3abc \qquad + b^3 + c^3 & & &
\end{array}
$$

In the first line of the work we multiply by a, and in the second by $(b+c)$, noting that

$$(b+c)^2 = b^2 + 2bc + c^2 \text{ and } (b^2 - bc + c^2)(b+c) = b^3 + c^3.$$

129. Expansion of Powers of x+a. The following table shows the coefficients of the terms in the expansion of powers of the binomial $x+a$.

$x+a$	1,	1.					
$(x+a)^2$	1,	2,	1.				
$(x+a)^3$	1,	3,	3,	1.			
$(x+a)^4$	1,	4,	6,	4,	1.		
$(x+a)^5$	1,	5,	10,	10,	5,	1.	
$(x+a)^6$	1,	6,	15,	20,	15,	6,	1.

It will be observed that *any number in the above table is obtained by adding the number immediately above it and the number to the left of the one immediately above.*

Thus above 20 is 10, to the left of 10 is 10, and $20 = 10 + 10$.

That the numbers in the above table are connected by the relation just mentioned will appear from the next example:

Ex. 1. *Obtain the expansion of* $(x+a)^4$ *from that of* $(x+a)^3$.

$$\begin{array}{rl} (x+a)^3 = & x^3 + 3x^2a + 3xa^2 + a^3 \\ & x + a \\ \hline & 1 + 3 + 3 + 1 \\ & \quad\ \ 1 + 3 + 3 + 1 \\ \hline (x+a)^4 = & x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4 \end{array}$$

Here each line of detached coefficients is the same, but the lower row is displaced one step to the right.

Ex. 2. *Obtain the expansion of* $(x+a)^7$ *from that of* $(x+a)^6$.

The coefficients in $(x+a)^7$ are obtained as below:

$$\begin{array}{l|llllllll} (x+a)^6 & 1 \rightarrow & 6 \rightarrow & 15 \rightarrow & 20 \rightarrow & 15 \rightarrow & 6 \rightarrow & 1. & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ (x+a)^7 & 1, & 7, & 21, & 35, & 35, & 21, & 7, & 1. \end{array}$$

The arrows indicate the way in which the coefficients are found; thus $1+6=7$, $6+15=21$, etc.

Again, $(x+a)^7$ is homogeneous and of the seventh degree in x and a and the coefficients just obtained belong to x^7, x^6a, x^5a^2, x^4a^3, etc.; so that

$$(x+a)^7 = x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7.$$

Ex. 3. *Write down the expansion of* $(2x-3)^4$.

$$\begin{aligned} (2x-3)^4 &= (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 \\ &= 16x^4 - 4.8.3x^3 + 6.4.9x^2 - 4.2.27x + 81 \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81. \end{aligned}$$

130. Picking out Coefficients. It often happens that only certain coefficients in the product of two given expressions are required. In such cases it is unnecessary to find the complete product.

Much labour will be saved by remembering that

(i) *The product of two expressions A and B is obtained by multiplying each term of A by every term of B and adding the results.*

(ii) *In finding a particular coefficient in a product, we are concerned only with those terms which give rise to this coefficient.*

Ex. 1. *Find the coefficient of x^3 in*

$$(1+2x+3x^2+4x^3)(2-3x+4x^2-5x^3).$$

The term containing x^3 in the product is obtained by multiplying 1 by $(-5x^3)$, $(2x)$ by $(4x^2)$, $(3x^2)$ by $(-3x)$, $(4x^3)$ by 2 and adding the results.

$$\begin{aligned}\therefore \text{ coefficient of } x^3 &= 1\,.\,(-5)+2\,.\,4+3(-3)+4\,.\,2\\ &= -5+8-9+8\\ &= 2.\end{aligned}$$

Ex. 2. *Find the coefficient of x^4 in $(1-x^2+x^4)^3$.*

Write y for x^2-x^4, then

$$\begin{aligned}(1-x^2+x^4)^3 &= (1-y)^3\\ &= 1-3y+3y^2-y^3\\ &= 1-3(x^2-x^4)+3(x^2-x^4)^2-(x^2-x^4)^3.\end{aligned}$$

The coefficient of x^4 in $-3(x^2-x^4)$ is 3.

The coefficient of x^4 in $3(x^2-x^4)^2$ is 3

$$(\text{for } (x^2-x^4)^2 = x^4-2x^6+x^8).$$

The coefficient of x^4 in $-(x^2-x^4)^3$ is 0 (for x^6 is the lowest power of x in $(x^2-x^4)^3$).

$\therefore$ the coefficient of x^4 in the given expression is $3+3+0$ or 6.

EXERCISE XLVII.

Expand the following products :

1. $(3a^2-5ab+2b^2)(a^2-7ab)$.
2. $(x^2-2x+5)(x^2+x-3)$.
3. $(x^2-3x+7)(2x^2+5x-3)$.
4. $(x^2-7xy+8y^2)(-3x^2+4xy-2y^2)$.
5. $(2x^2-5x+3)(7x^2+x-8)$.
6. $(x^4-ax^3-a^3x+a^4)(x^2-a^2)$.
7. $(a^5+a^4b-ab^4-b^5)(a^2b-ab^2)$.
8. $(x^3-7x+6)(x^2-5x+2)$.
9. $(5x^3-xy^2-7y^3)(x^2-4xy+3y^2)$.
10. $(3x^3-4x+8)(-1+2x+x^2)$.
11. $(5-11x^2+x^3)(2x^2+7x-3)$.
12. $(x+1)(x+2)(x+3)(x+4)$.
13. $(2x+1)(3x-2)(4x+3)(5x-4)$.
14. $(2x-3)^2(3x-1)^2$.
15. $(2x^3-x^2+x-2)(3x^2+x-5)$.
16. $(2a^2b^2-4ab^3+5a^3b)(5a^2-2ab-4b^2)$.
17. $(1+2x+3x^2+4x^3)(1-x)^2$.
18. $(x^4+x^3-x-1)(x^2-x+1)$.
19. $(x^4+2x^3-8x-16)(x^2-2x+4)$.
20. $(x^4+2x^3+3x^2+2x+1)(x-1)^2$.
21. $(2+4x+6x^2+3x^3)(1-x)^2$.

22. $(2x^3-3x^2+x-11)(x^3-x^2-2x+3)$.

23. $(3-2x-3x^2+5x^3)(2-3x^2-5x^3)$.

24. $(5x^3+2x^2-3x-7)(2x^3-3x^2+2x-5)$.

25. $(a+x)(b-x)(c+x)$. 26. $(x^2+y^2+z^2-yz+zx+xy)(x-y-z)$.

27. $(4x^2+9y^2+z^2+3yz-2zx+6xy)(2x-3y+z)$.

28. $(x^3-ax^2-bx+c)(x^2-x-1)$.

Write down the expansions of the following :

29. $(x-1)^4$. 30. $(1-x)^5$. 31. $(x-y)^6$.

32. $(2x+1)^4$. 33. $(2-3x)^5$. 34. $(1+2x)^6$.

35. Find the coefficient of x^3 in the product of $3x^3-4x^2+5x-6$ and $4x^3-2x^2-x-4$.

36. Find the coefficient of x^6 in the product of

$$(1-2x+4x^2-8x^3+16x^4)(1+2x+4x^2+8x^3+16x^4).$$

37. Find the coefficient of x^3 in the expansions of the following :
(i) $(3+2x)^3(1-x)$; (ii) $(2-3x)^2(1+5x)^2$; (iii) $(1-x+x^2)^2$;
(iv) $(1+x+x^2)^3$; (v) $(1+2x+3x^2+4x^3)^2$.

38. Find the coefficient of x^4 in the expansion of $(1-x)^6(1+x+2x^2)$.

39. Find the coefficient of x^2 in $(x+a)^2(x+b)^2-(a+b)^2(x+a+b)^2$.

40. If the coefficients of x^4 and x in the product

$$(2x^3+3x^2+ax-10)(3x^3-ax^2-10x+4)$$

are equal to one another, find the value of a.

131. The two Meanings of "Division." In arithmetic the word "division" has two distinct meanings which are as follows :

1. Division is the inverse of multiplication, and to divide a by b is to find the number which when multiplied by b produces a.

2. Division is a process of continued subtraction, and to divide 17 by 5 is to answer the following questions :

(i) How many times must 5 be subtracted from 17 that the remainder may be less than 5 ? (ii) what is this remainder ?

Now $$17=5+5+5+2=5\times 3+2.$$

Thus if 5 is subtracted 3 times from 17, the remainder is 2.

The meaning of **division as continued subtraction** in arithmetic is then as follows:

To divide a number N by a number D, which is less than N, is to find numbers Q and R such that

$$N = DQ + R,$$

where R is less than D.

If N and D stand for given numbers, the values of Q and R can be found by continued subtraction (as in the instance given above), and each has one value and one only.

The number N is called the **Dividend**, D the **Divisor**, Q the **Quotient** and R the **Remainder.**

In the important special case *when R is zero, N is exactly divisible by D, and the two meanings of "division" coincide.*

132. Long Division. In the last article N and D represented known numbers: now let these letters stand for polynomials in some letter x, and let the degree of N be not lower than that of D. It is *possible* that N may be *exactly divisible* by D (that is to say D *may* be a factor of N). We shall describe a process of continued subtraction, called **Long Division**, whereby we can always discover if N *is* exactly divisible by D.

Take two particular instances:—

(1) Let $\quad N = 6x^2 + 19x + 15; \quad D = 3x + 5.$

In the process *both* expressions must be arranged in descending powers of x or *both* in ascending powers. We take them arranged in descending powers and proceed as follows:

Divide the left-hand term of N by the left-hand term of D; the quotient is $2x$. Subtract $2xD$ from N; thus:

$$\begin{aligned} N &= 6x^2 + 19x + 15 \\ 2xD &= 6x^2 + 10x \\ \hline N - 2xD &= \qquad\quad 9x + 15 \end{aligned}$$

Divide the left-hand term of the remainder by the left-hand term of D; the quotient is 3. Subtract $3D$ from the remainder; thus:

$$\begin{aligned} N - 2xD &= 9x + 15 \\ 3D &= 9x + 15 \\ \hline N - 2xD - 3D &= \quad 0 \end{aligned}$$

The final remainder is zero. This remainder has been obtained by subtracting $D(2x+3)$ from N; hence

$$N = D(2x+3) \quad \text{or} \quad 6x^2+19x+15 = (3x+5)(2x+3).$$

The process is usually arranged as follows:

$$\begin{array}{r@{\,}l}
3x+5\ ) & 6x^2+19x+15\ (\ 2x+3 \\
& \underline{6x^2+10x} \\
& \qquad\quad 9x+15 \\
& \qquad\quad \underline{9x+15}
\end{array}$$

$$\therefore\ 6x^2+19x+15 = (3x+5)(2x+3).$$

(2) Let
$$N = 6x^4+2x^3-3x^2+7x+10,$$
$$D = 3x^2+4x-5.$$

Both N and D being arranged in descending powers of x, the process is as follows:

$$\begin{array}{r@{\,}l}
3x^2+4x-5\ ) & 6x^4+2x^3-\ \ 3x^2+7x+10\ (\ 2x^2-2x+5 \\
& \underline{6x^4+8x^3-10x^2} \\
& \quad\ -6x^3+\ \ 7x^2+\ \ 7x \\
& \quad\ \underline{-6x^3-\ \ 8x^2+10x} \\
& \qquad\qquad\ 15x^2-\ \ 3x+10 \\
& \qquad\qquad\ \underline{15x^2+20x-25} \\
& \qquad\qquad\qquad\ -23x+35
\end{array}$$

$$\therefore\ N = D(2x^2-2x+5)+(-23x+35).$$

In the first step we divide $6x^4$ by $3x^2$; the quotient is $2x^2$. We subtract $2x^2(3x^2+4x-5)$ from N, and the remainder is

$$-6x^3+7x^2+7x+10.$$

The term $+10$ is not "brought down" till later.

We next divide $(-6x^3)$ by $3x^2$; the quotient is $(-2x)$, and we subtract $(-2x)(3x^2+4x-5)$ from the first remainder. The second remainder is $15x^2-3x+10$. We divide $15x^2$ by $3x^2$; the quotient is 5, and we subtract $5(3x^2+4x-5)$ from the second remainder.

The third remainder $(-23x+35)$ is of *lower degree than D, and the process cannot be continued any further.* This last remainder is the result of subtracting $D(2x^2-2x+5)$ from N, so that N is not divisible by D, but the process leads to the identity

$$N = D(2x^2-2x+5)+(-23x+35).$$

The process just described is called **long division**, and, *although no division actually takes place*, it is customary to say that N has been *divided* by D and to call $2x^2-2x+5$ the *quotient* (Q) and $(-23x+35)$ the *remainder* (R). The result of the process is *to express* N *in the form* $DQ+R$, where R is of *lower degree than* D.

This result is called the **Division Transformation.** When (later) we employ fractional coefficients, it will be seen that:

If D is not of higher degree than N, this transformation can always be effected in one way and in one way only.

133. Arrangement in Ascending Powers. In the last article, the expressions denoted by N and D were arranged in descending powers of x: if they are arranged in ascending powers, the process of long division is as follows:

(1) Let $N=15+19x+6x^2$; $D=5+3x$.

$$
\begin{array}{r|l}
5+3x\,) & 15+19x+6x^2\,(\,3+2x \\
 & \underline{15+\ \ 9x} \\
 & \qquad 10x+6x^2 \\
 & \qquad \underline{10x+6x^2}
\end{array}
$$

$$\therefore\ 15+19x+6x^2=(5+3x)(3+2x).$$

Thus N is divisible by D, and the result is the same as that obtained by arranging N and D in descending powers.

(2) Let $N=10+7x-3x^2+2x^3+6x^4$;
$D=-5+4x+3x^2$.

$$
\begin{array}{r|l}
-5+4x+3x^2\,) & 10+\ 7x-\ 3x^2+\ 2x^3+\ 6x^4\,(\,-2-3x-3x^2 \\
 & \underline{10-\ 8x-\ 6x^2} \\
 & \qquad 15x+\ 3x^2+\ 2x^3 \\
 & \qquad \underline{15x-12x^2-\ 9x^3} \\
 & \qquad\qquad 15x^2+11x^3+\ 6x^4 \\
 & \qquad\qquad \underline{15x^2-12x^3-\ 9x^4} \\
 & \qquad\qquad\qquad 23x^3+15x^4
\end{array}
$$

$$\therefore\ N=D(-2-3x-3x^2)+(23x^3+15x^4).$$

Here N is not exactly divisible by D, and the identity resulting from

this process of long division is totally different from that obtained by arranging N and D in descending powers.

Observe that the degree of every term of the remainder $23x^3+15x^4$ is higher than that of any term of the quotient $(-2-3x-3x^2)$, and it will be seen later that by employing fractional coefficients *the process of division can be continued indefinitely so as to obtain a quotient with any number of terms.*

If then N and D are polynomials arranged in ascending powers of some letter x, the process of long division enables us to find polynomials Q and R such that

$$\mathbf{N = DQ + R},$$

where Q has any number of terms, and the degree of the lowest term in R is higher than that of the highest term in Q

Ex. *Divide 1 by $1-x$ so as to obtain a quotient of four terms and the corresponding remainder. State the result of the division in the form of an identity.*

$$\begin{array}{rlllll}
1-x\,) & 1 & & & & (\,1+x+x^2+x^3 \\
 & \underline{1} & \underline{-x} & & & \\
 & & x & & & \\
 & & \underline{x} & \underline{-x^2} & & \\
 & & & x^2 & & \\
 & & & \underline{x^2} & \underline{-x^3} & \\
 & & & & x^3 & \\
 & & & & \underline{x^3} & \underline{-x^4} \\
 & & & & & x^4
\end{array}$$

$\therefore$ Quotient $=1+x+x^2+x^3$, Remainder $=x^4$.

The resulting identity is

$$1=(1-x)(1+x+x^2+x^3)+x^4.$$

134. Method of Detached Coefficients.—The labour connected with "long division" can be shortened by *detaching the coefficients*, at the same time keeping each in its proper relative position. In the following example divisions already performed in Arts. 132, 133 are worked out in this manner.

Ex. *If* $N=6x^4+2x^3-3x^2+7x+10$, $D=3x^2+4x-5$, *find the quotient and remainder when* N *is divided by* D: (i) *when* N *and* D *are arranged in descending powers of* x, *and* (ii) *when they are arranged in ascending powers: in the latter case find a quotient of three terms.*

(i)

$$\begin{array}{r|rrrrrl}
3+4-5\,) & 6 & +2 & -3 & +7 & +10 & (\,2-2+5 \\
 & 6 & +8 & -10 & & & \\ \cline{2-4}
 & & -6 & +7 & +7 & & \\
 & & -6 & -8 & +10 & & \\ \cline{3-5}
 & & & 15 & -3 & +10 & \\
 & & & 15 & +20 & -25 & \\ \cline{4-6}
 & & & & -23 & +35 &
\end{array}$$

Supplying the letters, we have

Quotient $=2x^2-2x+5$, Remainder $=-23x+35$.

(ii)

$$\begin{array}{r|rrrrrl}
-5+4+3\,) & 10 & +7 & -3 & +2 & +6 & (\,-2-3-3 \\
 & 10 & -8 & -6 & & & \\ \cline{2-4}
 & & 15 & +3 & +2 & & \\
 & & 15 & -12 & -9 & & \\ \cline{3-5}
 & & & 15 & +11 & +6 & \\
 & & & 15 & -12 & -9 & \\ \cline{4-6}
 & & & & 23 & +15 &
\end{array}$$

$\therefore$ Quotient $=-2-3x-3x^2$, Remainder $=23x^3+15x^4$.

In using this method, the following points must be noticed:

(i) The coefficients belonging to the same power of x should be written in a vertical line: we are thus able to supply the letters correctly in the remainder.

(ii) The left-hand term of the quotient is obtained by dividing the left-hand term of N by the left-hand term of D. This enables us to supply the letters in the quotient.

(iii) If any terms are *missing* from N or from D, these must be supplied with zero coefficients. Thus, to divide x^4-1 by x^2-1, we divide $x^4+0x^3+0x^2+0x-1$ by x^2+0x-1, commencing thus:

$$1+0-1\,)\,1+0+0+0-1\,($$

If any coefficients of the quotient are zero, these should be written down; in the case just considered the quotient is given by $1+0+1$, and is therefore x^2+1.

EXERCISE XLVIII.

In each of the examples 1-32, apply the process of long division to show that the first expression is exactly divisible by the second and to find the quotient.

1. $x^2+9x+20$; $x+4$.

2. $x^2+5x-14$; $x+7$.

3. $x^2-11x+24$; $x-3$.

4. $6x^2+31x+35$; $3x+5$.

5. $15x^2-22x-48$; $5x+6$.

6. $21x^2-13xy-18y^2$; $3x+2y$.

7. x^3+1; $x+1$.

8. x^4-1; $x+1$.

9. x^5+1; $x+1$.

10. x^6+y^6; x^2+y^2.

11. x^5-y^5; $x-y$.

12. $27x^3-6x-4$; $3x-2$.

13. $2x^3+4x^2+8x+16$; $x+2$.

14. $2x^3-3x^2-19x-15$; $2x+3$.

15. $2x^4-5x^3-6x^2+19x-10$; $2x-5$.

16. $2x^4+x^3y+x^2y^2+xy^3-y^4$; $2x-y$.

17. $12x^4-17x^3y+19xy^3-10y^4$; $3x-2y$.

18. $x^5-x^3+3x^2-6x-9$; x^2-3.

19. $x^4-4x^2-12x-9$; x^2+2x+3.

20. $2x^4-9x^3+4x^2-25$; $2x^2-3x+5$.

21. $81x^4+36x^2+16$; $9x^2-6x+4$.

22. $2x^4+27y^4-5x^3y-24x^2y^2$; x^2-9y^2-3xy.

23. $1+3x-24x^2+8x^4$; $2x^2+3x-1$.

24. $x^4-56x+15$; $x^2+4x+15$.

25. $x^5+5x^4-9x^3-38x^2-x+6$; x^3-7x-3.

26. $5x^4-4x^3+3x^2+22x+55$; x^2-3x+5.

27. $2x^4-7x^3y-12x^2y^2-27y^4$; $x^2-3xy-9y^2$.

28. $8x^5-12x^3+35x^2+16x-7$; $2x^2+3x-1$.

29. $12x^5+11x^4-25x^3+14x^2-2x-4$; $4x^2-3x+2$.

30. $3x^5-5x^4-48x+80$; $x-10+3x^2$.

31. $10x^5+x^4+31x^2+8x-39$; $13-7x+5x^2$.

32. $6x^5-17x^4+42x^3-66x^2+72x-72$; $2x^2-3x+6$.

In each of the Examples 33-36, find the value of a for which the first expression is exactly divisible by the second.

33. x^3+2x^2+a; $x-1$.

34. $2x^4-a$; $x+1$.

35. $2x^3-9x^2+5x+a$; $2x-3$.

36. $4+7x^2+ax^4$; $2+x$.

In each of the Examples 37–41, find the expression of lowest degree in x which must be subtracted from the first expression that the result may be exactly divisible by the second expression.

37. $4x^4+9x^3-35x^2+44x-10$; x^2+4x-3.

38. $15x^4-x^3+2x^2+37x-70$; $3x^2-2x+7$.

39. $x^6-4x^3+2x^2-5x-8$; x^2-x+2.

40. $x^4+2x^3-15x^2-15x+32$; x^2+3x-5.

41. $-1+x-2x^2+3x^3-4x^4+5x^5$; $3-2x+x^2$.

In Examples 42 and 43, find the quotient and remainder when N is divided by D (i) when N and D are arranged in descending powers of x ; (ii) when N and D are arranged in ascending powers of x : in this case find a quotient of three terms.

42. $N=6x^4+11x^3+9x^2+17x+10$; $D=3x^2+4x-5$.

43. $N=8x^4-6x^3-33x^2-49x-15$; $D=2x^2-3x-5$.

In Examples 44–51, given that the second polynomial is a factor of the first, express the first polynomial as the product of linear factors

44. $x^3+2x^2-29x+42$; $x-2$.

45. $x^3-6x^2+11x-6$; $x-1$.

46. $x^3-6x^2-7x+60$; $x-5$.

47. $12x^3-16x^2-81x-35$; $3x+5$.

48. $6x^4-x^3-27x^2-2x+24$; $3x^2-2x-8$.

49. $60x^4-83x^3-84x^2+17x+6$; $4x^2-5x-6$.

50. $x^5+x^4-2x^3-2x^2+x+1$; $(x+1)^3$.

51. $6x^5+7x^4-94x^3+113x^2+28x-60$; $(x-1)(x-2)(x+5)$.

PART III. RATIONAL NUMBERS

CHAPTER XVII.

FRACTIONS.

135. Fractions. In order that a value of x may exist which satisfies the equation $xb = a$, when a is not divisible by b, we invent a new class of numbers called fractions.

When a is not divisible by b, the symbol $\frac{a}{b}$ denotes a **fraction** whose numerator is a and **denominator** b. The fraction $\frac{1}{a}$ is called the **reciprocal** of a.

When a is divisible by b, by the Rule of Signs,

$$\frac{-a}{b} = -\frac{a}{b}; \quad \frac{a}{-b} = -\frac{a}{b}; \quad \frac{-a}{-b} = \frac{a}{b}.$$

When a is not divisible by b, these equations will be assumed to hold, and will enable us to assign a value to a negative fraction.

It has been explained that the *value* of an integer depends on its place on the scale; it will be shown in Art. 136 that the system of numbers consisting of integers (positive and negative, including zero) and fractions *can be arranged in a definite order* and the **value of a fraction** will be defined by its place in the scale so formed.

136. Definitions of Equality and Inequality. In connection with two integers denoted by a and b, the terms 'greater than,' 'equal to' and 'less than' have been defined as referring to the relative position of a and b on the scale

$$\ldots -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ldots.$$

We proceed to extend the ideas of equality and inequality to fractions.

In Art. 71 it has been shown that if a is divisible by b and x is divisible by y, then $\frac{a}{b} >, =$ or $< \frac{x}{y}$, according as $ay >, =$ or $< xb$. If then the rules of equality and inequality are to be the same for fractions as for integers, we are led to the following rule, which in the case of fractions amounts to a *definition*:

Rule. If a, x, stand for any integers, whether positive, zero or negative, and if b, y stand for any positive integers, then $\frac{a}{b} >, =$ or $< \frac{x}{y}$, according as $ay >, =$ or $< xb$.

This rule assigns to every fraction a definite place in the scale

$$\ldots -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3 \ldots.$$

Thus (i) $\frac{5}{6} > \frac{3}{4}$, for $5 \times 4 > 3 \times 6$; again, $-\frac{13}{2}$ is to be placed between -6 and -7 on the scale, for

$$-13 < -12 \text{ and } -13 > -14.$$

(ii) If $x = 0$ and a, b, y are positive, the rule asserts that $\frac{a}{b} > \frac{0}{y}$ if $ay > 0 . b$. Now $ay > 0$, for a and y are positive, $\therefore \frac{a}{b} > 0$; hence *every positive fraction is greater than zero.*

Again, $-\frac{a}{b} = \frac{-a}{b}$ and $\frac{-a}{b} < \frac{0}{y}$, for $-ay < 0$; hence *every negative fraction is less than zero.*

137. The Rational System of Numbers. The system of numbers comprising positive and negative integers (including zero) and positive and negative fractions is called the **Rational System of Numbers**, and every number of this system is said to be **rational**.

The **Rational Scale** is constructed by placing the rational numbers in the order determined by the rule of Art. 136.

138. Fundamental Operations with Fractions. If a and b stand for natural numbers, $a + b$ has been defined as the bth number after a on the natural scale, and $a \times b$ has been defined as the sum of b numbers, each equal to a.

These definitions assign no meaning to such expressions as $\frac{2}{3} + \frac{4}{5}$ and $\frac{2}{3} \times \frac{4}{5}$.

As in the case of negative numbers, we *invent* new operations with fractions which are called addition, subtraction, multiplication and division.

Addition and Subtraction. If both a and b are divisible by c, it has been shown that

$$\text{(i)}\ \frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}. \qquad \text{(ii)}\ \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

These formulae are taken to **define** *addition and subtraction when one or both of the expressions* $\frac{a}{c}$, $\frac{b}{c}$ *are fractions.* It is evident that addition and subtraction continue to be inverse operations, and to obey the commutative and associative laws.

Multiplication and Division. If a is divisible by b and x is divisible by y, it has been shown that

$$\text{(iii)}\ \frac{a}{b}\cdot\frac{x}{y}=\frac{ax}{by}. \qquad \text{(iv)}\ \frac{a}{b}\div\frac{x}{y}=\frac{ay}{bx}.$$

In case one or both of the expressions $\frac{a}{b}$, $\frac{x}{y}$ *are fractions, these formulae will be taken to* **define** *multiplication and division.* It is evident that multiplication and division continue to be inverse operations, and it can be shown that they conform to the commutative, associative and distributive laws.

We may now *define* $\frac{a}{b}$ by the equation $\left(\frac{a}{b}\right)b=a$, for

$$\left(\frac{a}{b}\right)b=\frac{a}{b}\cdot\frac{b}{1}=\frac{ab}{b}=a.$$

139. Reduction of a Fraction to Lowest Terms.

Theorem. If $\frac{a}{b}$ is a fraction, then

$$\text{(i)}\ \frac{ax}{bx}=\frac{a}{b}. \qquad \text{(ii)}\ \frac{\frac{a}{x}}{\frac{b}{x}}=\frac{a}{b}.$$

Proof. By the laws of multiplication and division for integers,

$$(ax)b=a(bx) \quad\text{and}\quad \left(\frac{a}{x}\right)b=\left(\frac{b}{x}\right)a.$$

Hence it follows from the definition of equality for fractions that

$$\frac{ax}{bx}=\frac{a}{b} \quad\text{and}\quad \frac{\frac{a}{x}}{\frac{b}{x}}=\frac{a}{b}. \qquad \textit{(See Art. 136)}$$

Hence **a fraction is unaltered if its numerator and denominator are both multiplied, or both divided, by the same number.**

The fraction $\frac{a}{b}$ is said to be expressed in its **lowest terms** if a and b have no common factor except unity.

Rule. *To express a fraction in its lowest terms, divide the numerator and denominator by the* H.C.F. *of the numerator and denominator.*

140. Rules for Addition and Subtraction of Fractions.

$$\text{(i)}\quad \mathbf{\frac{a}{b}+\frac{x}{y}=\frac{ay}{by}+\frac{bx}{by}=\frac{ay+bx}{by}}.$$

$$\text{(ii)}\quad \mathbf{\frac{a}{b}-\frac{x}{y}=\frac{ay}{by}-\frac{bx}{by}=\frac{ay-bx}{by}}.$$

141. Index Laws.

The results of Art. 68 are:

(i) If m is greater than n, then $\frac{x^m}{x^n}=x^{m-n}$;

(ii) If x is divisible by y, then $\left(\frac{x}{y}\right)^n=\frac{x^n}{y^n}$.

If m is less than n, dividing numerator and denominator of the fraction $\frac{x^m}{x^n}$ by x^m we have

$$\frac{x^m}{x^n}=\frac{1}{x^{n-m}}. \quad\text{.............................(iii)}$$

If x is not divisible by y, it follows from the definition of multiplication for fractions that

$$\left(\frac{x}{y}\right)^n=\frac{x^n}{y^n}, \quad\text{.............................(iv)}$$

for $\quad \left(\frac{x}{y}\right)^2=\frac{x}{y}\cdot\frac{x}{y}=\frac{x^2}{y^2};\quad \left(\frac{x}{y}\right)^3=\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}=\frac{x^2}{y^2}\cdot\frac{x}{y}=\frac{x^3}{y^3},$

and the formula (iv) can be proved for the values 4, 5, 6 ... of n in succession.

142. Permanence of Algebraic Form.

Since all rational numbers are subject to the same Rules of Equality, and combine with one another according to the same fundamental laws, it follows that *every formula which is true when the letters stand for natural numbers is also true for all rational values of the letters.*

143. Examples on Substitution.

Ex. *If $2a = -3$ and $3b = 4$, find the values of*

$$\text{(i) } \frac{a}{a-b}; \quad \text{(ii) } a^3b^2; \quad \text{(iii) } \sqrt{10(a+b)(2a+b)}.$$

Since $2a = -3$ and $3b = 4$, $\therefore\ a = -\frac{3}{2}$, $b = \frac{4}{3}$; hence

$$\text{(i)} \quad \frac{a}{a-b} = \frac{-\frac{3}{2}}{-\frac{3}{2}-\frac{4}{3}} = \frac{-\frac{3}{2}}{-\frac{17}{6}} = \frac{3}{2} \cdot \frac{6}{17} = \frac{9}{17};$$

$$\text{(ii)} \quad a^3b^2 = \left(-\frac{3}{2}\right)^3 \cdot \left(\frac{4}{3}\right)^2 = -\frac{3^3 \cdot 2^4}{2^3 \cdot 3^2} = -3 \cdot 2 = -6;$$

$$\text{(iii)} \quad a+b = -\frac{3}{2}+\frac{4}{3} = \frac{-9+8}{6} = -\frac{1}{6},$$

$$2a+b = -3+\frac{4}{3} = \frac{-9+4}{3} = -\frac{5}{3};$$

$$\therefore \quad \sqrt{10(a+b)(2a+b)} = \sqrt{10\left(-\frac{1}{6}\right)\left(-\frac{5}{3}\right)} = \sqrt{\frac{5^2}{3^2}} = \frac{5}{3}.$$

EXERCISE XLIX.

1. If $x = -\frac{1}{3}$, $y = -1$, $z = 3$, find the value of $x^5y^3z^6$.

2. If $a = 1$, $b = -2$, $c = 3$, and $d = 0$, find the value of the expression

$$\text{(i) } \frac{b^2c^2}{a} + \frac{a^2c^2}{b} - \frac{b^2d^2}{c}; \qquad \text{(ii) } \frac{bc+ca-bd}{b+c+d}.$$

3. If $3a = -2$ and $4b = 3$, find the numerical values of

$$(a)\ \frac{b}{a-b} + \frac{2a}{b-a}; \quad (b)\ \sqrt[3]{\frac{4a^2}{a-\frac{b}{9}}}; \quad (c)\ \sqrt{\left(5 - \frac{19}{8} \cdot \frac{a}{b}\right)}.$$

4. Find the sum of $a+2b-3c$, $-3a+b+2c$ and $2a-3b+2c$. Verify your answer when $a = \frac{1}{5}$, $b = \frac{1}{3}$, $c = \frac{1}{4}$.

5. If $a = 5$, $b = -1$, find the value of

$$\text{(i) } \sqrt{\frac{a+b}{3a-b}} - \frac{a+10b}{a-b}; \qquad \text{(ii) } \sqrt{\left\{\left(\frac{a}{a+b}\right)^2 + (a+2b)^2\right\}}.$$

6. If $3x=1,\ 2y=-3,\ z=-4$, find the values of

$$(a)\ 6xy-4yz+2zx;\quad (b)\ \frac{9x}{8y^2}+\frac{24y}{z^2}+\frac{z}{36x^2};\quad (c)\ \frac{3x}{y+z}+\frac{z}{x+y}.$$

7. If $2x=3,\ 4y=3$ and $z=-2$, find the values of

$$(a)\ \sqrt{(8y+2z+7)}+\sqrt{(6x-8y+z)};\quad (b)\ \frac{x}{y}+\frac{4x+2z}{2x-z}.$$

8. If $2a=1$ and $b=-2$, find the value of

$$\text{(i)}\ \frac{3a+4b}{6a-5b}+\sqrt{\frac{4a}{a-2b}};\qquad \text{(ii)}\ \sqrt{\left\{b^2+(a+b)^2+\left(\frac{b}{a+b}\right)^2\right\}}.$$

9. If $a=7$ and $x=-16$, find the value of

$$\text{(i)}\ \frac{a+x}{a-x}+\sqrt{\frac{a+x}{a+2x}};\qquad \text{(ii)}\ \sqrt{\frac{1}{2}(a+1)^2+\frac{1}{2}(a+x)^2-\left(\frac{a-x-10}{2a+x}\right)^2}.$$

10. If $x=-1,\ y=-2,\ z=\frac{1}{2}$, find the values of

$$(a)\ 2x-\{9y-8x+2z-(4x+y)\};$$
$$(b)\ (x+y-z)^2+(x+y)^2(x-y+z)+(x-y)^3.$$

11. If $2x=1,\ 3y=-1,\ 5z=1$, find the value of

$$\text{(i)}\ \frac{1}{x+y}+\frac{1}{x+z}+\frac{1}{y+z};\qquad \text{(ii)}\ \sqrt[3]{(x^2-y^2)(x+2y)^2(z+1)}.$$

12. If $x=2,\ y=4,\ z=5$, find the value of

$$\text{(i)}\ \frac{y-z}{y+z-2x}-\frac{z-x}{z+x-2y}+\frac{x+y}{x+y-2z};\qquad \text{(ii)}\ \sqrt[3]{\left(\frac{1}{xy}\right)^3+(z-y)^3+\left(\frac{1-zx}{2y}\right)^3}.$$

13. If $2a=-3$ and $5b=1$, find the values of

$$\text{(i)}\ \frac{3b}{a-b}-\frac{2(b^2-4ab)}{(b^2-a^2)};\quad \text{(ii)}\ \sqrt{a^2-b^2-1}-\sqrt{\frac{3ab(2a-b)}{5(3b-2a)}}.$$

14. If $a=3,\ b=5,\ c=7$, find the value of

$$\frac{a}{(b-c)(c-a)}-\frac{b}{(c-a)(a-b)}+\frac{c}{(a-b)(b-c)}.$$

15. Given $u=25{\cdot}24$, $v=13{\cdot}27$, calculate x to four significant figures from the equation

$$\frac{1}{x}=\frac{1}{u}+\frac{1}{v}.$$

16. From the equation

$$t=2\pi\sqrt{(l\div g)}$$

find l in terms of the other quantities, and calculate its value to three significant figures when $t=1$, $g=32{\cdot}18$, $\pi=3{\cdot}1416$.

CHAPTER XVIII.

SIMPLIFICATION OF FRACTIONS.

144. Examples on Reduction of a Fraction to Lowest Terms.

Ex. *Reduce to their lowest terms the fractions*

$$\text{(i)}\ \frac{(2a^2b-4ab^2)^2}{2a^3b-8ab^3};\quad \text{(ii)}\ \frac{2x^2+3xy-2y^2}{7xy-2x^2-3y^2}.$$

$$\text{(i)}\ \frac{(2a^2b-4ab^2)^2}{2a^3b-8ab^3}=\frac{4a^2b^2(a-2b)^2}{2ab(a^2-4b^2)}=\frac{2ab(a-2b)}{a+2b}.$$

$$\text{(ii)}\ \frac{2x^2+3xy-2y^2}{7xy-2x^2-3y^2}=-\frac{2x^2+3xy-2y^2}{2x^2-7xy+3y^2}=-\frac{(2x-y)(x+2y)}{(2x-y)(x-3y)}$$

$$=-\frac{x+2y}{x-3y}.$$

EXERCISE L.

Simplify the following:

1. $\dfrac{(4x^2+2xy)^2}{8x^2+8xy+2y^2}$. **2.** $\dfrac{9x^2+12x+4}{(9x^2+6x)^2}$. **3.** $\dfrac{2x^2-xy-y^2}{4x^2-3xy-y^2}$.

4. $\dfrac{x^2-17x+70}{x^2-2x-35}$. **5.** $\dfrac{6x^2-49x+65}{14x^2-93x+13}$. **6.** $\dfrac{2x^2-xy-y^2}{3xy-x^2-2y^2}$.

7. $\dfrac{17xy-x^2-70y^2}{x^2-2xy-35y^2}$. **8.** $\dfrac{3abc-3b^2c}{a-bc-b(1-c)}$. **9.** $\dfrac{(y-x)^2(x^2+xy+y^2)}{(x^2-y^2)(x^3-y^3)}$.

Simplify the following :

10. $\dfrac{(x^2+3xy+2y^2)(y^2-3xy+2x^2)}{(x^2-y^2).(x^2-4y^2)}$.

11. $\dfrac{xy+4ab+2ay+2bx}{xy-4ab+2ay-2bx}$.

12. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x+1}$.

13. $\dfrac{x^3+2x^2-x-2}{x^3+3x^2-4x-12}$.

14. $\dfrac{3abc-3b^2c}{a^2-abc-ab(1-c)}$.

15. $\dfrac{x^5-x^4y-xy^4+y^5}{x^4-x^3y-x^2y^2+xy^3}$.

16. $\dfrac{4x^4-5x^2+1}{4x^4-6x^3+2x^2}$.

17. $\dfrac{x^4-7x^2+1}{(x^2+1)^2+4x(x^2+1)+3x^2}$.

18. $\dfrac{(x^2+6x+11)^2-9(x+3)^2}{(x^2+7x+9)^2-(2x+5)^2}$.

19. $\dfrac{(12x^2+5x-2)^2-(23x+10)^2}{(8x^2+4x-15)^2-(4x+17)^2}$.

20. $\dfrac{x^2-7x-60}{x^2+17x+60}\times\dfrac{1}{x^2-144}$.

21. $\dfrac{x^2-5x+6}{x^2-4}\times\dfrac{2x^2-9x+4}{2x^2-7x+3}$.

22. $\dfrac{a^2+ab}{a^2-ab}\times\dfrac{ab-b^2}{ab+b^2}\times\dfrac{ab-bc}{ab+bc}$.

23. $\dfrac{a^2-ab}{2bc-ab}\times\dfrac{b^2+bc}{ac-bc}\times\dfrac{2ac-bc}{ab+ac}$.

24. $\dfrac{(a+b+c)^2}{a^2-(b+c)^2}\times\dfrac{(a-b)^2-c^2}{(c+a)^2-b^2}$.

25. $\dfrac{x^2+2x-3}{x^2-6x+9}\div\dfrac{x^2+x-6}{x^2-5x+6}$.

26. $\dfrac{6x^2-ax-2a^2}{ax-a^2}\times\dfrac{x-a}{9x^2-4a^2}\div\dfrac{a+2x}{a(2a+3x)}$.

27. $\dfrac{a^2-4b^2}{b^2-4c^2}\times\dfrac{b^2+bc-2c^2}{a^2-ab-2b^2}\times\dfrac{b^2-bc-2c^2}{a^2+ab-2b^2}$.

28. $\dfrac{(x-y)^2-z^2}{xy-y^2-yz}\times\dfrac{z}{x^2+xy-xz}\div\dfrac{xz-yz+z^2}{x^2-(y-z)^2}$.

29. $\dfrac{2x^2+x-1}{x^2-4x+3}\times\dfrac{2x^2-5x+3}{6x^2+x-2}\times\dfrac{3x^2-7x-6}{2x^2-7x+6}$.

30. $\dfrac{x^4-8x}{x^2-4x-5}\times\dfrac{x^2+2x+1}{x^3-x^2-2x}\div\dfrac{x^2+2x+4}{x-5}$.

31. $\dfrac{a^3-1}{a^2+a-6}\times\dfrac{a^2-4a+4}{a^2-4a+3}\times\dfrac{a^2-9}{a^4+a^2+1}$.

32. $\dfrac{x^2-4}{x^2+3x+2}\times\dfrac{(x+1)^2}{x^2-x-12}\div\dfrac{x^2-x-2}{x^2-4x}$.

33. $\dfrac{x^3-y^3}{(x+y)^2}\times\dfrac{x^3+y^3}{x^2+xy-2y^2}\div\dfrac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}$.

34. $\dfrac{x^2-5x+6}{x^2+x-12}\times\dfrac{x^2-16}{x^3-8}\div\dfrac{(x-2)^2-x}{(x+2)^2-2x}$.

35. $\dfrac{x^3-a^3}{x^2-ax+bx-ab}\times\dfrac{x^2-b^2}{x^2+ax-bx-ab}\div\dfrac{x^2+ax+a^2}{x^2+ax+bx+ab}$.

145. Examples on the Addition of Fractions.

Ex. 1. *Simplify* $\dfrac{2}{1-9x^2}-\dfrac{1}{2(1+3x)}-\dfrac{1}{2(1-3x)}$.

The L.C.M. of the denominators is $2(1+3x)(1-3x)$ and

$$\text{the expression}=\frac{4-(1-3x)-(1+3x)}{2(1+3x)(1-3x)}$$

$$=\frac{2}{2(1+3x)(1-3x)}=\frac{1}{1-9x^2}.$$

Ex. 2. *Simplify* $\dfrac{5}{2x^2-7x+3}-\dfrac{1}{6x^2-5x+1}$.

$$\text{The expression}=\frac{5}{(2x-1)(x-3)}+\frac{1}{(2x-1)(3x-1)}$$

$$=\frac{5(3x-1)+(x-3)}{(2x-1)(x-3)(3x-1)}$$

$$=\frac{16x-8}{(2x-1)(x-3)(3x-1)}$$

$$=\frac{8(2x-1)}{(2x-1)(x-3)(3x-1)}$$

$$=\frac{8}{(x-3)(3x-1)}.$$

Ex. 3. *Simplify* $\dfrac{5x-12}{x^2-5x+6}-\dfrac{7x-24}{x^2-7x+12}+\dfrac{3x-4}{x^2-6x+8}$.

The expression

$$=\frac{5x-12}{(x-2)(x-3)}-\frac{7x-24}{(x-3)(x-4)}+\frac{3x-4}{(x-2)(x-4)}$$

$$=\frac{(5x-12)(x-4)-(7x-24)(x-2)+(3x-4)(x-3)}{(x-2)(x-3)(x-4)}$$

$$=\frac{5x^2-32x+48-7x^2+38x-48+3x^2-13x+12}{(x-2)(x-3)(x-4)}$$

$$=\frac{x^2-7x+12}{(x-2)(x-3)(x-4)}$$

$$=\frac{(x-3)(x-4)}{(x-2)(x-3)(x-4)}=\frac{1}{x-2}.$$

EXERCISE LI.

Simplify the following expressions :

1. $\dfrac{x^2}{x^2-1}-\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}$.

2. $\dfrac{2}{a-2b}+\dfrac{1}{a+2b}-\dfrac{4a}{a^2-4b^2}$.

3. $\dfrac{a-b}{2a+2b}+\dfrac{a+b}{2a-2b}-\dfrac{2ab}{a^2-b^2}$.

4. $3\dfrac{x-1}{x+1}-5\dfrac{x+1}{x-1}+2\cdot\dfrac{x^2+1}{x^2-1}$.

5. $\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}$.

6. $\dfrac{1}{x^2-9}-\dfrac{1}{x^2+x-6}$.

7. $\dfrac{1}{(x-3)(x+1)}-\dfrac{1}{(x-3)(x+1)(x-2)}$.

8. $\dfrac{2}{x^2-4x+3}-\dfrac{1}{x^2-3x+2}$.

9. $\dfrac{1}{(x-a)(x+b)}-\dfrac{a-b}{(x-a)(x+b)(x+a)}$.

10. $\dfrac{a}{c}-\dfrac{(a-bc)x}{c(c+x)}-\dfrac{a}{c+x}$.

11. $\dfrac{1}{(a+b)(a+2b)}-\dfrac{b}{(a+b)(a+2b)(a+3b)}$.

12. $\dfrac{x^2-x+1}{x^2(x+1)}+\dfrac{2}{x}-\dfrac{2}{x+1}$.

13. $\dfrac{2}{x^2-ax}-\dfrac{3}{x^2-a^2}+\dfrac{4}{x^2+ax}$.

14. $\dfrac{5}{(2x-7)(4x-9)}-\dfrac{6}{(2x-7)(10x-29)}$.

15. $\dfrac{1}{2x^2-3ax-2a^2}-\dfrac{1}{2x^2-5ax+2a^2}$.

16. $\dfrac{x^2-6x+8}{x^2-x-12}-\dfrac{x^2+3x-10}{x^2+8x+15}$.

17. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}$.

18. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-4\dfrac{y}{x}$.

19. $\dfrac{4}{x-3a}-\dfrac{x-5a}{x^2-5ax+6a^2}-\dfrac{3}{x-2a}$.

20. $\dfrac{2}{x^2-1}+\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-7x+12}$.

21. $\dfrac{1}{x^2-5x+4}-\dfrac{2}{x^2-3x-4}+\dfrac{3}{x^2-1}$.

22. $\dfrac{3x+2}{3x^2-x-2}-\dfrac{3x-2}{3x^2+x-2}-\dfrac{2}{x^2-1}$.

23. $\dfrac{1}{x^2-3x+2}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-5x+4}$.

24. $\dfrac{1}{2a^2-5ab+2b^2}-\dfrac{2}{2a^2-3ab-2b^2}+\dfrac{1}{2a^2+3ab-2b^2}$.

25. $\dfrac{x-3}{(x-4)(x-5)}+\dfrac{2(4-x)}{(x-3)(x-5)}+\dfrac{x-5}{(x-3)(x-4)}$.

26. $\dfrac{x^2}{(x+1)(x-1)(x-2)}-\dfrac{1}{6(x+1)}-\dfrac{4}{3(x-2)}$.

27. $\dfrac{5}{2}-\dfrac{x}{x+1}+\dfrac{x}{x-4}-\dfrac{2x^2-3x-10}{x^2-3x-4}$.

28. $\dfrac{x+4}{x(x-2)}+\dfrac{x+2}{x(x-4)}+\dfrac{5-2x}{x^2-6x+8}$.

29. $\dfrac{a+b-c}{a-b+c}-\dfrac{a-b+c}{a+b-c}-\dfrac{4(b-c)^2}{a^2-(b-c)^2}$.

30. $\dfrac{x-y}{x^3-y^3}-\dfrac{x+y}{x^3+y^3}+\dfrac{2(x^2+y^2)}{x^4+x^2y^2+y^4}$.

31. $\dfrac{2}{x^2-12x+35}-\dfrac{3}{x^2-9x+20}+\dfrac{2}{x^2-8x+15}$.

32. $\dfrac{x+1}{x^2-5x+6}-\dfrac{8}{x^2-4x+3}+\dfrac{3}{x^2-3x+2}$.

33. $\dfrac{5x}{2x^2-7x+3}-\dfrac{6}{4x^2-8x+3}-\dfrac{3x}{2x^2-9x+9}$.

34. $\dfrac{x}{2x^2-5x+3}+\dfrac{5x-4}{3x^2-5x+2}-\dfrac{x-4}{6x^2-13x+6}$.

35. $\dfrac{x+1}{x^2+x-6}+\dfrac{2x-1}{x^2+2x-3}-\dfrac{3x+1}{x^2-3x+2}$.

36. $\dfrac{3x+11}{x^2-x-6}-\dfrac{x-7}{x^2-7x+12}-\dfrac{2x-2}{x^2-6x+8}$.

37. $\dfrac{x^2}{x^2+3x+2}+\dfrac{x+6}{x^2+5x+6}-\dfrac{6}{x^2+4x+3}$.

38. $\dfrac{2(x+1)}{2x^2-3x}-\dfrac{x^2-1}{x^3+x}-\dfrac{5x-1}{2x^4-3x^3+2x^2-3x}$.

146. Addition of Fractions (*continued*).

It is generally advisable to *arrange all the numerators and all the denominators of fractions which are to be added in ascending (or descending) powers of some letter.*

Ex. 1. *Simplify* $\dfrac{1}{a+b}+\dfrac{3b}{ab-a^2}+\dfrac{4b-a}{a^2-b^2}$.

If we arrange the numerators and denominators in descending powers of a, noting that

$$ab-a^2=-(a^2-ab)=-a(a-b) \text{ and } 4b-a=-(a-4b),$$

the given expression $=\dfrac{1}{a+b}-\dfrac{3b}{a(a-b)}-\dfrac{a-4b}{(a+b)(a-b)}$ (*Art. 135*)

$$=\frac{a(a-b)-3b(a+b)-a(a-4b)}{a(a+b)(a-b)}$$

$$=\frac{a^2-ab-3ab-3b^2-a^2+4ab}{a(a+b)(a-b)}$$

$$=\frac{-3b^2}{a(a^2-b^2)}=\frac{3b^2}{a(b^2-a^2)}. \quad (\textit{Art. 135})$$

In many cases the work is shortened by *adding the fractions in groups.*

Ex. 2. *Simplify* $\dfrac{1}{a}-\dfrac{3}{a+2}+\dfrac{3}{a+4}-\dfrac{1}{a+6}$.

$$\text{The given expression}=\frac{1}{a}-\frac{1}{a+6}-3\left\{\frac{1}{a+2}-\frac{1}{a+4}\right\}$$

$$=\frac{6}{a(a+6)}-3\cdot\frac{2}{(a+2)(a+4)}$$

$$=6\left\{\frac{1}{a(a+6)}-\frac{1}{(a+2)(a+4)}\right\}$$

$$=6\cdot\frac{(a+2)(a+4)-a(a+6)}{a(a+6)(a+2)(a+4)}$$

$$=6\cdot\frac{a^2+6a+8-a^2-6a}{a(a+6)(a+2)(a+4)}$$

$$=\frac{48}{a(a+2)(a+4)(a+6)}.$$

EXERCISE LII.

Simplify the following expressions :

1. $\dfrac{1}{a}+\dfrac{1}{a-b}-\dfrac{b}{ab-a^2}$.

2. $\dfrac{6a}{a^2-4}-\dfrac{2}{2+a}+\dfrac{3}{2-a}$.

3. $\dfrac{3}{1+a}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}$.

4. $\dfrac{2b-a}{ab+b^2}-\dfrac{2a+b}{a^2-ab}+\dfrac{a-b}{ab}$.

5. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}$.

6. $a+\dfrac{x^2+a^2}{2x-a}+\dfrac{x^2+2ax}{a-2x}$.

7. $\dfrac{pq}{ab}+\dfrac{(p-b)(q-b)}{b(b-a)}+\dfrac{(p-a)(q-a)}{a(a-b)}$.

8. $\dfrac{1}{x+a}-\dfrac{2}{x}+\dfrac{1}{x-a}$.

9. $\dfrac{1}{x^2-1}+\dfrac{2}{3+2x-x^2}+\dfrac{1}{x^2-4x+3}$.

10. $\dfrac{1}{x+2}-\dfrac{2}{x+1}+\dfrac{1}{x}$.

11. $\dfrac{1}{a^2-3a+2}-\dfrac{5}{a^2+a-6}+\dfrac{4}{2a-3+a^2}$.

12. $\dfrac{1}{x+2}-\dfrac{3}{x+3}+\dfrac{3}{x+4}-\dfrac{1}{x+5}$.

13. $\dfrac{a}{(a-x)^2}-\dfrac{3a}{2a^2-ax-x^2}+\dfrac{1}{2a+x}$.

14. $\dfrac{1}{x+4}-\dfrac{3}{x+2}+\dfrac{3}{x}-\dfrac{1}{x-2}$.

15. $\dfrac{1}{abx}+\dfrac{1}{a(b-a)(a-x)}-\dfrac{1}{b(a-b)(x-b)}$.

16. $\dfrac{3}{3-x}+\dfrac{1}{x+5}+\dfrac{2}{x-7}$.

17. $\dfrac{1-x}{2-x}-\dfrac{x-2}{x-1}+\dfrac{3}{(x-1)(x-2)}$.

18. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}$.

19. $\dfrac{2x-1}{x-1}-\dfrac{4x-6}{x-2}+\dfrac{2x-5}{x-3}$.

20. $\dfrac{2a+b}{a-2b}+\dfrac{2a-b}{a+2b}+1-\dfrac{40a^2b^2}{a^4-16b^4}$.

21. $\dfrac{a+b}{a-b}-\dfrac{2ab}{a^2-b^2}-\dfrac{2a^2b^2}{a^4-b^4}$.

22. $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}$.

23. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{x-2}{x^2-x+1}-\dfrac{x+2}{x^2+x+1}$.

24. $\dfrac{x+1}{x^2-x}-\dfrac{x+2}{x^2-1}-\dfrac{1}{x^3+1}$.

25. $\dfrac{3a-5}{(2a-7)(a+2)}-\dfrac{3-4a}{(2+a)(5-3a)}+\dfrac{5a-12}{(3a-5)(2a-7)}$.

26. $\dfrac{2a-1}{(a-1)(2a-3)}+\dfrac{8(1-a)}{(1-2a)(3-2a)}+\dfrac{2a-3}{(2a-1)(a-1)}$.

27. $\dfrac{x-1}{6x^2-7x+2}-\dfrac{1-x}{12x^2-17x+6}+\dfrac{1}{8x^2-10x+3}$.

28. $\dfrac{9x+14}{x^2-16}+\dfrac{4-5x}{x^2-5x+4}-\dfrac{4x+5}{x^2+3x-4}$.

29. $\dfrac{2x+1}{x^2-x-2}-\dfrac{3x-2}{2x^2+3x+1}-\dfrac{3(5x-1)}{2+3x-2x^2}$.

147. Miscellaneous Fractions.

Ex. 1. *Simplify* $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{y}{x}-\dfrac{x}{y}}$.

First method. Multiplying numerator and denominator by xy,

$$\text{the expression}=\frac{y-x}{y^2-x^2}=\frac{1}{x+y}.$$

Second method.

$$\frac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{y}{x}-\dfrac{x}{y}}=\frac{\dfrac{y-x}{xy}}{\dfrac{y^2-x^2}{xy}}=\frac{y-x}{xy}\cdot\frac{xy}{y^2-x^2}=\frac{1}{x+y}.$$

Ex. 2. *Simplify* $\dfrac{\dfrac{a+b}{1-ab}-\dfrac{a+c}{1-ac}}{1+\dfrac{a+b}{1-ab}\cdot\dfrac{a+c}{1-ac}}$.

Multiplying numerator and denominator by $(1-ab)(1-ac)$,

$$\text{the expression} = \frac{(a+b)(1-ac)-(a+c)(1-ab)}{(1-ab)(1-ac)+(a+b)(a+c)}$$

$$= \frac{a+b-a^2c-abc-a-c+a^2b+abc}{1-ab-ac+a^2bc+a^2+ab+ac+bc}$$

$$= \frac{b-c+a^2b-a^2c}{1+a^2+bc+a^2bc}$$

$$= \frac{(b-c)(1+a^2)}{(1+a^2)(1+bc)} = \frac{b-c}{1+bc}.$$

Ex. 3. *Simplify* $\dfrac{x}{1-\dfrac{x}{2+x-\dfrac{4x}{3+2x}}}$.

An expression of this kind is called a *continued fraction*: to effect the simplification we work *upwards from the bottom.* The first step is to multiply numerator and denominator of

$$\frac{x}{2+x-\dfrac{4x}{3+2x}}$$

by $3+2x$, and the work is arranged as follows:

$$\text{The expression} = \frac{x}{1-\dfrac{x(3+2x)}{(2+x)(3+2x)-4x}}$$

$$= \frac{x}{1-\dfrac{3x+2x^2}{6+3x+2x^2}}$$

$$= \frac{x(6+3x+2x^2)}{6+3x+2x^2-3x-2x^2}$$

$$= \frac{6x+3x^2+2x^3}{6} = x+\frac{x^2}{2}+\frac{x^3}{3}.$$

EXERCISE LIII.

Simplify the following expressions :

1. $\dfrac{1}{1+\dfrac{x}{y}}+\dfrac{1}{1+\dfrac{y}{x}}$.

2. $\dfrac{1}{1-\dfrac{x}{2y}}+\dfrac{1}{1-\dfrac{2y}{x}}$.

3. $\dfrac{x}{1-\dfrac{x}{y}}-\dfrac{y}{1-\dfrac{y}{x}}-\dfrac{2}{\dfrac{1}{x}-\dfrac{1}{y}}$.

4. $\dfrac{x-2-\dfrac{3}{x}}{1-\dfrac{1}{x^2}}$.

5. $\dfrac{x-\dfrac{2}{x}-1}{x-\dfrac{6}{x}+1}$.

6. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$.

7. $\dfrac{x^3-a^3}{x^2-a^2}\div\left(x+\dfrac{a^2}{x}+a\right)$.

8. $\left\{\dfrac{1+x}{1-x}-\dfrac{1-y}{1+y}\right\}\div\left\{1+\dfrac{(1+x)(1-y)}{(1-x)(1+y)}\right\}$.

9. $\dfrac{1}{1+\dfrac{1}{a^2}}\times\dfrac{1}{a^2-\dfrac{1}{a^2}}\div\dfrac{1}{1+\dfrac{2a^2}{a^4+1}}$.

10. $\dfrac{\dfrac{x^3-y^3}{x-y}-\dfrac{x^3+y^3}{x+y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$.

11. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}\div\dfrac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}}-\dfrac{\dfrac{2}{x}}{1+\dfrac{1}{x^2}}$.

12. $\left(\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{b}-\dfrac{1}{a}}\right)(a+b)+\left(\dfrac{\dfrac{b}{a}-\dfrac{a}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}\right)(a-b)$.

13. $\dfrac{\dfrac{x}{x+1}+\dfrac{x}{x-1}}{\dfrac{2}{x^2-1}}\quad\dfrac{4x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$.

14. $\dfrac{1+\dfrac{b-c}{b+c}}{1-\dfrac{b-c}{b+c}}\times\dfrac{1+\dfrac{c-a}{c+a}}{1-\dfrac{c-a}{c+a}}\times\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}}$.

15. $\dfrac{1}{x-\dfrac{x}{x+1}}+\dfrac{1}{x+\dfrac{x}{x-1}}+\dfrac{1-\dfrac{3}{x}}{1-\dfrac{x}{3}}$.

16. $\dfrac{\dfrac{x-y}{1+xy}+\dfrac{y-z}{1+yz}}{1-\dfrac{(x-y)(y-z)}{(1+xy)(1+yz)}}$.

17. $\dfrac{a}{b-\dfrac{a}{b-\dfrac{a}{b}}}$.

18. $\dfrac{1+x}{1-\dfrac{1-x}{1-\dfrac{1}{x}}}$.

19. $\dfrac{x-3}{x-3-\dfrac{x}{x+\dfrac{3x+1}{x-3}}}$.

20. $\dfrac{x-2}{x-2-\dfrac{x^2-2x}{x-\dfrac{2}{x}}}$.

21. $\dfrac{x}{1+\dfrac{x}{2-x+\dfrac{4x}{3-2x}}}$.

22. $\dfrac{19x+4}{4+\dfrac{3}{4+\dfrac{1}{x}}}-\dfrac{13x-4}{4-\dfrac{3}{4-\dfrac{1}{x}}}$.

23. Substitute $\dfrac{2a-9b}{4a-21b}$ for x in $\dfrac{a-9bx}{ax-b}$, and reduce the result to its simplest form.

24. Substitute $\dfrac{a-1}{a+1}$ for x in each of the following expressions, and reduce the results to their simplest forms:

(i) $\dfrac{1}{1-x}+\dfrac{1}{1+x}$. (ii) $\dfrac{2-3x}{4+5x}$.

25. Substitute $\dfrac{a-1}{a+1}$ for x and $\dfrac{2a-1}{2a+1}$ for y and the expression $\dfrac{x-y}{x+y}$, and reduce the result to its simplest form.

EXERCISE LIV.

MISCELLANEOUS EXERCISE ON FRACTIONS.

Simplify the following expressions:

1. $\dfrac{1}{(1+x)^2}-\dfrac{2}{1-x^2}+\dfrac{1}{(1-x)^2}$.

2. $\left(\dfrac{1}{x+3}-\dfrac{1}{2x+3}\right)\left(2+\dfrac{3}{x}\right)$.

3. $\left(\dfrac{3}{2a-b}-\dfrac{3}{3a+b}\right)\left(1+\dfrac{2a}{a+b}\right)$.

4. $\dfrac{1}{x-1}+\left(\dfrac{x+\frac{1}{2}}{x-1}-\dfrac{x+1}{x-\frac{1}{2}}\right)$.

5. $\left(\dfrac{1}{x+1}+\dfrac{1}{x+5}\right)\left(\dfrac{3}{x-1}-\dfrac{1}{x+3}\right)$.

6. $\left(\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}+1\right)$.

7. $\left\{1-\dfrac{x^2+y^2}{2xy}\right\}\left\{1-\dfrac{(x+y)^2}{(x-y)^2}\right\}$.

8. $\dfrac{x^2y^2+2xy}{x^2y-2xy^2}\times\dfrac{xy+2}{x-2y}\div\dfrac{x^2y^2+4xy+4}{x^2-4xy+4y^2}$.

9. $\dfrac{x^2-(y-z)^2}{(x+z)^2-y^2}+\dfrac{y^2-(z-x)^2}{(y+x)^2-z^2}+\dfrac{z^2-(x-y)^2}{(z+y)^2-x^2}$.

10. $\dfrac{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}{1-\dfrac{a^2+b^2}{(a-b)^2}}$.

11. $\dfrac{\dfrac{x+2}{x+1}-\dfrac{1}{x+3}}{\dfrac{1}{x+1}+\dfrac{x+2}{x+3}}$.

12. $\left\{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right\}\div\left\{\dfrac{a^2+ax}{ax-x^2}-\dfrac{a^2-ax}{ax+x^2}\right\}$.

13. $\dfrac{(1+x)^2-x(1+x)}{(1-x)^2+x(1-x)}\div\dfrac{(1+x)^2-x(1+x)^2}{(1-x)^2+x(1-x)^2}$.

14. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{c}}\times\dfrac{\dfrac{1}{a+b}-\dfrac{1}{c}}{\dfrac{1}{a-c}+\dfrac{1}{b}}$.

15. $\left\{\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}\right\}\times\left\{x+\dfrac{3}{x-4}\right\}$.

16. $\dfrac{a-b+\dfrac{ab}{a-b}}{a-\dfrac{b(a-b)}{a}} \times \left(\dfrac{a}{b}-\dfrac{b}{a}\right).$ 17. $\dfrac{x^2+a^2}{2a^2+3ax-2x^2} \div \left(\dfrac{2a+x}{2a-x}+\dfrac{a-2x}{a+2x}\right).$

18. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right) \div \left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right).$

19. $\left(\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}\right) \times \left(\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}\right).$

20. $\left\{\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right\} \times \left\{\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right\}.$

21. $\left(1+\dfrac{3}{x+3}-\dfrac{3}{x-1}\right) \times \left(2-\dfrac{3}{x+5}+\dfrac{3}{x-3}\right).$

22. $\left\{\dfrac{a}{a+b}-\dfrac{b}{a-b}-\dfrac{2b^2}{b^2-a^2}\right\} \div \left\{1-\dfrac{2b}{a+b}\right\}^2.$ 23. $\dfrac{3}{4}-\dfrac{3}{4+\dfrac{1}{x+6}}.$

24. $\left(a-\dfrac{x-2a}{a-1}\right)\left(1+\dfrac{x-2a}{a-1}\right)+\left(\dfrac{x-2a}{a-1}\right)^2.$

25. $\left(1-\dfrac{1}{1+x}-\dfrac{x}{1-x^2}+\dfrac{x^2}{1+x^3}\right) \times \dfrac{x-1}{x}.$

26. $\left(\dfrac{a^3}{b}-\dfrac{b^3}{a}\right)\left(\dfrac{3a+b}{a+b}-\dfrac{3a-b}{a-b}\right).$ 27. $\left(\dfrac{1}{1+x}+\dfrac{2x}{1-x^2}\right) \times \dfrac{x^3+2x-3}{x^2+x+3}.$

28. $\dfrac{(a+b)^3-(a-b)^3}{(a+b)^3+(a-b)^3} \times \dfrac{\dfrac{1}{a^2+b^2}+\dfrac{1}{2b^2}}{\dfrac{1}{a^2+b^2}+\dfrac{1}{2a^2}}.$

29. $\left(\dfrac{a^2}{b}+\dfrac{b^2}{a}\right)\left(\dfrac{1}{b^2-a^2}\right)-\dfrac{b}{a^2+ab}+\dfrac{a}{ab-b^2}.$

30. $\left(\dfrac{b-c}{a+b-c}-\dfrac{a-b+c}{c-b}\right)\left(\dfrac{1}{a}-\dfrac{c-b}{a^2}\right).$

31. $\left(\dfrac{a}{a-b}-1\right) \div \dfrac{b^2}{a^2-b^2}+\left(\dfrac{a^3}{b^3}-1\right)\left(\dfrac{a^2+ab}{a^2+ab+b^2}-1\right).$

32. $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}} \div \dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{b}+\dfrac{1}{a}}.$ 33. $\dfrac{2y}{x+y}+\left(\dfrac{1+\dfrac{x}{y}+\dfrac{x^2}{y^2}}{1-\dfrac{x}{y}+\dfrac{x^2}{y^2}} \times \dfrac{y^3+x^3}{y^3-x^3}\right)+\dfrac{2y}{x-y}.$

34. $\dfrac{x^2+ax-2a^2}{x^2-(a+2b)x+2ab}-\dfrac{x^2-ax-2a^2}{x^2+(a+2b)x+2ab}.$

35. $\dfrac{x(x+y)+y^2}{x^2-y(x-y)} \times \dfrac{x^2(x-y)+y(x^2+y^2)}{x(x^2+y^2)-y^2(x+y)}$.

36. $\left(\dfrac{1}{x^3-y^3}-\dfrac{1}{x^3+y^3}\right) \div \left(\dfrac{1}{x-y}-\dfrac{1}{x+y}-\dfrac{2y}{x^2+y^2}\right)$.

37. $\dfrac{x+y}{x+2y+\dfrac{3y^2}{x-2y}} \times \dfrac{x-y}{x^2-xy-2y^2}$.

38. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}} \div \dfrac{x^4+x^2y^2+y^4}{(y-x)^2}$.

39. $\dfrac{a-\dfrac{b-a}{ab-1}}{a+\dfrac{c-a}{1-ac}} \times \dfrac{\dfrac{1}{a}-\dfrac{c-a}{ac-1}}{\dfrac{1}{a}+\dfrac{b-a}{1-ab}}$.

40. $\left(a+b-\dfrac{a^2+b^2}{a+b-\dfrac{ab}{a+b}}\right) \div \left(\dfrac{a^2-b^2}{a^3-b^3}\right)$.

41. $\left(1-\dfrac{2ab}{a^2+b^2}\right) \times \dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}} \div \left(a-\dfrac{2ab}{a+b}\right)$.

42. $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{3}{x}} \times \dfrac{\dfrac{x}{2}-1}{\dfrac{x}{2}+1} - \dfrac{1-\dfrac{2}{x}}{1+\dfrac{3}{x}} \div \dfrac{\dfrac{x}{2}-1}{\dfrac{x}{2}+1}$.

43. Find the value of $\dfrac{\dfrac{1}{x}+\dfrac{1}{a}}{\dfrac{2}{x}-\dfrac{1}{a}}$, when for x is substituted the value given by $x(a-b)=a^2$.

44. If $a=\dfrac{1}{1-b}$, $b=\dfrac{1}{1-c}$, prove that $c=\dfrac{1}{1-a}$.

45. If $x+\dfrac{1}{y}=1$ and $y-\dfrac{1}{z}=1$, prove that $xyz=1$.

46. If $x=1+\dfrac{1}{y}$, $y=1+\dfrac{1}{z}$, $z=1+\dfrac{1}{a}$, express the product xyz as simply as possible in terms of a.

47. If $u=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)$ and $v=\dfrac{1}{2}\left(x-\dfrac{1}{x}\right)$, find in its simplest form in terms of x the value of $u^2v^2+(u^2-1)(v^2+1)$.

48. Prove that the sum of the three fractions

$$\frac{b-c}{1+bc},\quad \frac{c-a}{1+ca},\quad \frac{a-b}{1+ab}$$

is equal to their product.

CHAPTER XIX.

LONG MULTIPLICATION AND THE DIVISION TRANSFORMATION (*Continued*).

148. Products involving Fractions. The student should now have acquired some facility in choosing the shortest method in multiplying algebraical expressions. The following are miscellaneous examples in multiplication.

Ex. 1. *Expand the product* $\left(\frac{2x}{3y}+1-\frac{3y}{2x}\right)\left(\frac{2x}{3y}-1-\frac{3y}{2x}\right)$.

$$\text{The product} = \left\{\left(\frac{2x}{3y}-\frac{3y}{2x}\right)+1\right\}\left\{\left(\frac{2x}{3y}-\frac{3y}{2x}\right)-1\right\}$$

$$=\left(\frac{2x}{3y}-\frac{3y}{2x}\right)^2-1$$

$$=\frac{4x^2}{9y^2}-2+\frac{9y^2}{4x^2}-1 \quad =\frac{4x^2}{9y^2}-3+\frac{9y^2}{4x^2}.$$

Ex. 2. *Expand* $\left(1+\frac{x}{2}-\frac{x^2}{3}\right)^2$.

We may proceed as in Art. 122, Ex. 3, or as follows :—

$$\text{The expression} = \left\{1+\left(\frac{x}{2}-\frac{x^2}{3}\right)\right\}^2$$

$$=1+2\left(\frac{x}{2}-\frac{x^2}{3}\right)+\left(\frac{x}{2}-\frac{x^2}{3}\right)^2$$

$$=1+x-\frac{2x^2}{3}+\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{9}$$

$$=1+x-\frac{5}{12}x^2-\frac{1}{3}x^3+\frac{1}{9}x^4.$$

Ex. 3. *Expand the product* $\left(\frac{x^2}{2}-\frac{x}{3}+\frac{1}{4}\right)\left(2x^2+\frac{x}{2}+\frac{1}{3}\right)$.

Detaching the coefficients,

$$\begin{array}{l} \frac{1}{2}-\frac{1}{3}+\frac{1}{4} \\ 2+\frac{1}{2}+\frac{1}{3} \\ \hline 1-\frac{2}{3}+\frac{1}{2} \\ \quad\;\; \frac{1}{4}-\frac{1}{6}+\frac{1}{8} \\ \qquad\quad \frac{1}{6}-\frac{1}{9}+\frac{1}{12} \\ \hline 1-\frac{5}{12}+\frac{1}{2}+\frac{1}{72}+\frac{1}{12} \end{array}$$

The product is of degree 4 ; and supplying the letters, we have

$$\text{Product} = x^4-\frac{5}{12}x^3+\frac{1}{2}x^2+\frac{1}{72}x+\frac{1}{12}.$$

EXERCISE LV.

Expand the following :

1. $\left(2x^2-\frac{1}{2}\right)\left(\frac{x}{2}-\frac{2}{x}\right)$.

2. $\left(x^2+\frac{x}{2}+\frac{2}{3}\right)\left(\frac{x}{2}-\frac{1}{4}\right)$.

3. $\left(\frac{x}{3}-\frac{1}{4}+\frac{1}{5x}\right)\left(\frac{x}{3}+\frac{1}{4}+\frac{1}{5x}\right)$.

4. $\left(\frac{2x}{3}-1+\frac{2}{5x}\right)\left(\frac{2x}{3}+1-\frac{2}{5x}\right)$.

5. $\left(1+\frac{2x}{3}-\frac{3x^2}{2}\right)^2$.

6. $\left(\frac{2x}{y}-1+\frac{y}{2x}\right)^2$.

7. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$.

8. $\left(\frac{3}{4}x^3-\frac{1}{2}x^2-\frac{1}{3}\right)\left(\frac{4}{3}x-2\right)$.

9. $\left(\frac{1}{4}x^2-\frac{1}{8}x+\frac{1}{9}\right)\left(\frac{3}{2}x^2+x+\frac{2}{3}\right)$.

10. $\left(\frac{x}{3}-2y+\frac{z}{2}\right)\left(\frac{x}{3}+2y+\frac{z}{2}\right)$.

11. $\left(\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2\right)\left(\frac{3}{2}x^2-\frac{3}{4}y^2\right)$.

12. $\left(x+5+\frac{25}{2x}\right)\left(x-5+\frac{25}{2x}\right)$.

13. $(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$.

14. $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{3}\right)\left(x-\frac{1}{6}\right)$.

15. $\left(x-\frac{2}{3}\right)^2\left(x+\frac{3}{2}\right)$.

16. $\left(x-\frac{y}{3}\right)^3\left(x+\frac{y}{3}\right)$.

17. $\left(x+\frac{1}{2}y+\frac{1}{3}z\right)\left(x^2+\frac{1}{4}y^2+\frac{1}{9}z^2-\frac{1}{6}yz-\frac{1}{3}zx-\frac{1}{2}xy\right)$.

18. $\left(x-\frac{2}{3}y+\frac{3}{2}z\right)\left(x^2+\frac{4}{9}y^2+\frac{9}{4}z^2+yz-\frac{3}{2}zx+\frac{2}{3}xy\right)$.

Find the coefficients of x^3 in each of the following:

19. $\left(\frac{x^3}{2}-\frac{x^2}{3}+\frac{x}{4}-\frac{1}{5}\right)\left(x^3+\frac{x^2}{2}+\frac{x}{3}+\frac{1}{4}\right)$.

20. $\left(\frac{x^3}{2}-\frac{x^2}{3}+\frac{x}{4}-\frac{1}{5}\right)^2$.

21. $\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)^3$.

22. $\left(x-\frac{1}{2}\right)^2\left(x+\frac{2}{3}\right)^2$.

23. $\left(x-\frac{1}{2}\right)^3\left(x+\frac{2}{3}\right)^2$.

149. Long Division. The following are examples of the process of "long division" in which some of the coefficients are fractions.

Ex. 1. *If $N=x^4-x^2+1$, $D=2x^2-x-3$, find the quotient and remainder when N is divided by D:* (i) *when N and D are arranged in descending powers of x; and* (ii) *when they are arranged in ascending powers: in the latter case, find a quotient of three terms.*

$$
\begin{array}{rl}
\text{(i)}\quad 2x^2-x-3\,\big) & x^4-\quad x^2+1\,\Big(\frac{1}{2}x^2+\frac{1}{4}x+\frac{3}{8} \\
& \underline{x^4-\frac{1}{2}x^3-\frac{3}{2}x^2} \\
& \qquad \frac{1}{2}x^3+\frac{1}{2}x^2+1 \\
& \qquad \underline{\frac{1}{2}x^3-\frac{1}{4}x^2-\frac{3}{4}x} \\
& \qquad\qquad \frac{3}{4}x^2+\frac{3}{4}x+1 \\
& \qquad\qquad \underline{\frac{3}{4}x^2-\frac{3}{8}x-\frac{9}{8}} \\
& \qquad\qquad\qquad \frac{9}{8}x+\frac{17}{8}
\end{array}
$$

$$\therefore\ \text{Quotient}=\frac{1}{2}x^2+\frac{1}{4}x+\frac{3}{8};\quad \text{Remainder}=\frac{9}{8}x+\frac{17}{8}.$$

The method is exactly the same as that in Art. 132; the left-hand term of N is divided by the left-hand term of D, giving $\frac{x^4}{2x^2}$ or $\frac{1}{2}x^2$; $\frac{1}{2}x^2D$ is subtracted from N. The left-hand term $\left(\frac{1}{2}x^3\right)$ of the remainder divided by $2x^2$ gives $\frac{1}{4}x$, which is the next term of the quotient.

(ii) We shall divide $1-x^2+x^4$ by $-3-x+2x^2$ by detaching the coefficients; supplying the missing terms with zero coefficients, we divide $1+0x-x^2+0x^3+x^4$ by $-3-x+2x^2$.

$$
\begin{array}{r|cccccl}
-3-1+2 & 1 & +0 & -1 & +0 & +1 & \Big(-\frac{1}{3}+\frac{1}{9}+\frac{2}{27} \\
 & 1 & +\frac{1}{3} & -\frac{2}{3} & & & \\
\cline{2-4}
 & & -\frac{1}{3} & -\frac{1}{3} & +0 & & \\
 & & -\frac{1}{3} & -\frac{1}{9} & +\frac{2}{9} & & \\
\cline{3-5}
 & & & -\frac{2}{9} & -\frac{2}{9} & +1 & \\
 & & & -\frac{2}{9} & -\frac{2}{27} & +\frac{4}{27} & \\
\cline{4-6}
 & & & & -\frac{4}{27} & +\frac{23}{27} &
\end{array}
$$

Supplying the letters as in Art. 134, we see that

$$\text{Quotient} = -\frac{1}{3}+\frac{1}{9}x+\frac{2}{27}x^2;$$

$$\text{Remainder} = -\frac{4}{27}x^3+\frac{23}{27}x^4.$$

Observe that when N and D are arranged in ascending powers of x the process of division can be continued so as to give a quotient with any number of terms and a remainder whose lowest term is of higher degree than the highest term of the quotient.

Ex. 2. *Continue the division in* Ex. 1 (i) *to obtain two more terms in the quotient.*

The final remainder in Ex. 1 (i) is $\frac{9}{8}x+\frac{17}{8}$, and we continue the division thus:

$$
\begin{array}{r|l}
2x^2-x-3 & \frac{9}{8}x+\frac{17}{8} \qquad \left(\frac{9}{16x}+\frac{43}{32x^2}\right. \\
 & \underline{\frac{9}{8}x-\frac{9}{16}-\frac{27}{16x}} \\
 & \qquad \frac{43}{16}+\frac{27}{16x} \\
 & \qquad \underline{\frac{43}{16}-\frac{43}{32x}-\frac{129}{32x^2}} \\
 & \qquad\qquad \frac{97}{32x}+\frac{129}{32x^2}
\end{array}
$$

Thus the quotient to 5 terms $=\frac{1}{2}x^2+\frac{1}{4}x+\frac{3}{8}+\frac{9}{16}\cdot\frac{1}{x}+\frac{43}{32}\cdot\frac{1}{x^2}$.

The corresponding remainder $=\frac{97}{32x}+\frac{129}{32x^2}$.

Observe that **when N and D are arranged in descending powers of x the process of division can be continued so as to obtain a quotient of any number of terms, and that after a certain term the quotient proceeds in powers of $\frac{1}{x}$.**

Ex. 3. *Prove the following identities:*

(i) $$\frac{x^4-x^2+1}{2x^2-x-3}=\frac{1}{2}x^2+\frac{1}{4}x+\frac{3}{8}+\frac{9}{16}\cdot\frac{1}{x}+\frac{43}{32}\cdot\frac{1}{x^2}+\frac{\frac{97}{32x}+\frac{129}{32x^2}}{2x^2-x-3}.$$

(ii) $$\frac{1-x^2+x^4}{-3-x+2x^2}=-\frac{1}{3}+\frac{1}{9}x+\frac{2}{27}x^2+\frac{-\frac{4}{27}x^3+\frac{23}{27}x^4}{-3-x+2x^2}.$$

These identities result from the divisions performed in examples 1 and 2.

Ex. 4. *Explain how to apply the identities in* Ex. 3 *to obtain an approximation to the value of* $\frac{x^4-x^2+1}{2x^2-x-3}$, *correct to two places of*

decimals: (i) when $x = 1000$; (ii) when $x = 0{\cdot}001$.

(i) If $\quad x = 1000, \quad \frac{1}{x} = 0{\cdot}001, \quad \frac{1}{x^2} = 0{\cdot}00001$, etc.;

and from identity (i) of Ex. 3 it follows that $\frac{1}{2}x^2 + \frac{1}{4}x + \frac{3}{8}$ is an approximation to the value of the given fraction, correct to two places of decimals.

$$\text{When } x = 1000, \quad \begin{aligned} \frac{1}{2}x^2 &= \frac{1}{2}.1000000 = 500000 \\ \frac{1}{4}x &= \frac{1}{4}.1000 \quad = \quad 250 \\ \frac{3}{8} &= \qquad\qquad {\cdot}375 \\ \hline \frac{1}{2}x^2 + \frac{1}{4}x + \frac{3}{8} &= 500250{\cdot}375 \end{aligned}$$

The required approximation is $500250{\cdot}38$.

(ii) When $x = 0{\cdot}001$, it follows from identity (ii) of Ex. 3 that $-\frac{1}{3}$ is the required approximation. The value of the given fraction correct to two places is therefore $-0{\cdot}33$.

150. Expressions containing several Letters. In the case of expressions (N and D) containing several letters, the process of long division may be employed to discover if one of the expressions (D) is a factor of the other (N).

We begin by choosing some letter which is common to N and D, and arranging both N and D in descending powers of this letter or both in ascending powers of the letter.

If D is not a factor of N, the identity resulting from the process will depend on the particular letter chosen.

Ex. 1. *Prove the identities:*

$$\frac{x^3 + x^2y + y^3}{x + y} = x^2 + \frac{y^3}{x + y} = y^2 - yx + 2x^2 - \frac{x^3}{y + x}.$$

These arise from "dividing" $x^3 + x^2y + y^3$ by $x + y$, arranging the expressions (i) in descending powers of x, and (ii) in descending powers of y.

Ex. 2. *Prove that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$, and find the quotient when the second expression is divided by the first.*

We choose one of the letters, say a, and arrange the expressions in descending powers of a.

$$
\begin{array}{r|l}
a+(b+c) & a^3 \qquad\qquad\qquad - 3abc + (b^3+c^3) \;\big(\, a^2 - a(b+c) + (b^2 - bc + c^2) \\
 & \underline{a^3 + a^2(b+c)} \\
 & \quad - a^2(b+c) - 3abc \\
 & \quad \underline{- a^2(b+c) - a(b+c)^2} \\
 & \qquad\qquad a(b^2 - bc + c^2) + (b^3+c^3) \\
 & \qquad\qquad \underline{a(b^2 - bc + c^2) + (b^3+c^3)}
\end{array}
$$

$$\therefore\ (a^3+b^3+c^3-3abc) \div (a+b+c) = a^2+b^2+c^2-bc-ca-ab.$$

EXERCISE LVI.

In Examples 1–22 apply the process of long division to show that in each case the second expression is a factor of the first and to find the quotient when the first is divided by the second.

1. $\frac{1}{2}x^3+\frac{1}{24}x+\frac{1}{12}$; $\frac{1}{2}x+\frac{1}{4}$.

2. $\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x-\frac{9}{16}$; $\frac{1}{2}x^2+\frac{3}{4}$.

3. $x^3-\frac{43}{24}x^2+\frac{1}{2}$; $\frac{3}{2}x-1$.

4. $\frac{1}{6}x^4-x^2+\frac{1}{3}x+\frac{2}{3}$; $\frac{1}{3}x-\frac{2}{3}$.

5. $x^4-\frac{13}{6}x^3+x^2+\frac{4}{9}x-\frac{2}{3}$; $\frac{4}{3}x-2$.

6. $\frac{3}{4}x^4-\frac{1}{2}x^3y+\frac{1}{4}xy^3-\frac{3}{16}y^4$; $\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2$.

7. $\frac{3x^2}{8}+\frac{1}{6}+\frac{2}{27x^2}$; $\frac{3x}{2}+1+\frac{2}{3x}$.

8. $x^2-\frac{4}{x^2}-x+\frac{4}{x}$; $x+\frac{2}{x}-1$.

9. $2x^2+\frac{9}{2x^2}-\frac{13}{2}$; $2x+\frac{3}{x}+5$.

10. $\frac{1}{6}x^4+\frac{1}{6}x^3-\frac{6}{25}x^2+\frac{13}{20}x-\frac{3}{8}$; $\frac{1}{2}x^2-\frac{2}{5}x+\frac{3}{4}$.

11. $x^4+\frac{5x^3}{6}-\frac{55x^2}{36}+\frac{5x}{6}-\frac{1}{6}$; $\frac{2}{3}+\frac{1}{x}-\frac{1}{2x^2}$.

12. $2a(a^3+b^3)+\frac{1}{15}b^2(a^2-b^2)-a^2b(a+15b)$; $2a^2+5ab-\frac{1}{3}b^2$.

13. $x^3-x^2(a+b+c)+x(bc+ca+ab)-abc$; $x-a$.

14. $x^3y+x^2(y^2+y-2)-2xy+2y+2$; $x+y+1$.

15. $8x^3-y^3+6xy+1$; $2x-y+1$.

16. $x^3+27a^3+9ax-1$; $x+3a-1$.

17. $1-x+y-xy-x^2y^2-x^2y^3+x^3y^2+x^3y^3$; $1-x+xy-x^2y$.

18. $9x^3-4xy^2-9x^2-9xy+2y^2+12x-5y-12$; $3x-2y-3$.

19. $8x^3-2(11a+b)x^2+(15a^2+8ab-3b^2)x-2ab(3a-b)$; $2x-3a+b$.

20. $9(a-b)^4-52(a-b)^2(c-d)^2+64(c-d)^4$; $3(a-b)-4(c-d)$.
[Write x for $(a-b)$ and y for $(c-d)$.]

21. $a^3-8b^3+c^3+4abc-2a^2b+4ab^2$; $a^2+4b^2+c^2+2bc-ac$.

22. $x^7+(a-b)x^5+(c-a)x^4+bx^3-a(a-b)x^2+(ab-b^2-ac)x+bc$;
$x^3+x(a-b)+c$.

Employ the process of long division to prove the following identities:

23. $\dfrac{1}{1-x}=1+x+x^2+x^3+\dfrac{x^4}{1-x}$.

24. $\dfrac{1}{x-1}=\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^4}+\dfrac{1}{x^4(x-1)}$.

25. $\dfrac{1}{2+3x}=\dfrac{1}{2}-\dfrac{3x}{4}+\dfrac{9x^2}{8}-\dfrac{27x^3}{8(2+3x)}$.

26. $\dfrac{1}{3x+2}=\dfrac{1}{3x}-\dfrac{2}{9x^2}+\dfrac{4}{27x^3}-\dfrac{8}{27x^3(3x+2)}$.

27. $\dfrac{1+x^2}{(1-x)^2}=1+2x+4x^2+6x^3+\dfrac{8x^4-6x^5}{(1-x)^2}$.

28. $\dfrac{x^2+1}{(x-1)^2}=1+\dfrac{2}{x}+\dfrac{4}{x^2}+\dfrac{6}{x^3}+\dfrac{8x-6}{x^3(x-1)^2}$.

29. $\dfrac{1+x}{(1-x)^3}=1+4x+9x^2+16x^3+\dfrac{25x^4-39x^5+16x^6}{(1-x)^3}$.

30. $\dfrac{x+1}{(x-1)^3}=\dfrac{1}{x^2}+\dfrac{4}{x^3}+\dfrac{9}{x^4}+\dfrac{16}{x^5}+\dfrac{25x^2-39x+16}{x^5(x-1)^3}$.

31. If $N=x^4+2x^3+3x^2+4x+7$ and $D=x^2-x+1$,
prove that $N=D^2+4D(x+1)+6x+2$.

32. Prove that $\dfrac{x^5-6x^2+5}{(x^2+1)^2}=x-\dfrac{2(x+3)}{x^2+1}+\dfrac{x+11}{(x^2+1)^2}$.

CHAPTER XX.

EQUAL AND UNEQUAL FRACTIONS.

151. Theorem on Equal Fractions. **If $\frac{x}{a}=\frac{y}{b}$, then each of these fractions is equal to $\frac{lx+my}{la+mb}$, where l and m have any values, positive or negative.**

Proof. Denote either of the given fractions by k, so that

$$\frac{x}{a}=\frac{y}{b}=k;$$

$$\therefore\ x=ka \text{ and } y=kb;$$

$$\therefore\ \frac{lx+my}{la+mb}=\frac{lka+mkb}{la+mb}=\frac{k(la+mb)}{la+mb}=k;$$

$$\therefore\ \frac{lx+my}{la+mb}=\frac{x}{a}=\frac{y}{b}.$$

Important special cases are (i) when $l=1$ and $m=1$; (ii) when $l=1$ and $m=-1$. These cases may be stated as follows:

$$\text{If } \frac{\mathbf{x}}{\mathbf{a}}=\frac{\mathbf{y}}{\mathbf{b}},\ \text{ then each fraction } = \frac{\mathbf{x+y}}{\mathbf{a+b}}=\frac{\mathbf{x-y}}{\mathbf{a-b}}.$$

Ex. 1. *If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.*

Let $\frac{a}{b}=k$, then $\frac{c}{d}=k$ and $a=kb$, $c=kd$;

$$\therefore\ \frac{a+b}{a-b}=\frac{kb+b}{kb-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1};$$

also
$$\frac{c+d}{c-d}=\frac{kd+d}{kd-d}=\frac{d(k+1)}{d(k-1)}=\frac{k+1}{k-1};$$

$$\therefore\ \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Ex. 2. *If* $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, *prove that each of these fractions is equal to* $\frac{lx+my+nz}{la+mb+nc}$, *where* l, m, n *have any values.*

[Put $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ and proceed as in the proof of the last theorem.]

Ex. 3. *If* $5x=6y$, *find the value of* $\frac{2x+y}{3x-y}$.

First method. Dividing each side of the equation $5x=6y$ by $5y$, we have $\frac{x}{y}=\frac{6}{5}$;

$$\therefore \frac{2x+y}{3x-y}=\frac{\frac{2x}{y}+1}{\frac{3x}{y}-1}=\frac{2\cdot\frac{6}{5}+1}{3\cdot\frac{6}{5}-1}=\frac{17}{13}.$$

Second method. Dividing each side of the equation $5x=6y$ by 5×6, we have $\frac{x}{6}=\frac{y}{5}$.

Let $\frac{x}{6}=k$; $\therefore \frac{y}{5}=k$ and $x=6k$, $y=5k$;

$$\therefore \frac{2x+y}{3x-y}=\frac{12k+5k}{18k-5k}=\frac{17k}{13k}=\frac{17}{13}.$$

Ex. 4. *If* $\frac{2x+y}{3x-y}=\frac{17}{13}$, *find the value of* $\frac{x}{y}$.

Multiplying each side of the given equation by $13(3x-y)$, we have

$$13(2x+y)=17(3x-y);$$
$$\therefore 26x+13y=51x-17y;$$
$$\therefore 30y=25x;$$
$$\therefore 6y=5x.$$

Dividing each side by $5y$, we have

$$\frac{x}{y}=\frac{6}{5}.$$

Ex. 5. *If* $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}$, *prove that* $x+y+z=0$.

Let $\dfrac{x}{b-c}=k$, then $\dfrac{y}{c-a}=k$ and $\dfrac{z}{a-b}=k$;

$$\therefore\ x=k(b-c),\quad y=k(c-a),\quad z=k(a-b);$$

$$\begin{aligned}\therefore\ x+y+z&=k(b-c)+k(c-a)+k(a-b)\\&=k(b-c+c-a+a-b)\\&=k\,.\,0\quad=0.\end{aligned}$$

Ex. 6. *If* $\dfrac{y-z}{b-c}=\dfrac{z-x}{c-a}$, *then* (i) *each fraction is equal to* $\dfrac{x-y}{a-b}$,

and (ii) $a(y-z)+b(z-x)+c(x-y)=0$.

For $$\frac{y-z}{b-c}=\frac{z-x}{c-a}=\frac{(y-z)+(z-x)}{(b-c)+(c-a)}=\frac{y-x}{b-a}=\frac{x-y}{a-b}$$

and $$\frac{y-z}{b-c}=\frac{z-x}{c-a}=\frac{x-y}{a-b}=\frac{a(y-z)+b(z-x)+c(x-y)}{a(b-c)+b(c-a)+c(a-b)}.$$

Since the denominator of the last fraction is zero, the numerator must also be zero in order that the fraction may have any (finite) value. (See Art. 111.)

Ex. 7. *If* $ax+by=c$ *and* $x+y=1$, *find the value of* $\dfrac{x}{y}$.

$$\because\ ax+by=c \text{ and } cx+cy=c,$$

$$\therefore\ x(c-a)+y(c-b)=0;$$

$$\therefore\ x(c-a)=y(b-c).$$

Dividing each side of this equation by $y(c-a)$,

$$\frac{x}{y}=\frac{b-c}{c-a}.$$

Ex. 8. *Apply the Theorem of Art. 151 to solve the simultaneous equations*

$$2x=3y \text{ and } 5x-6y=11.$$

Dividing each side of the first equation by 2×3, and applying the case of the above Theorem when $l=5$, $m=-6$, we have

$$\frac{x}{3}=\frac{y}{2}=\frac{5x-6y}{5\times 3-6\times 2}=\frac{11}{3};$$

$$\therefore\ x=11,\ y=\frac{22}{3}=7\tfrac{1}{3}.$$

The method employed in this example is very useful.

152. Theorem on Unequal Fractions. **If $\frac{x}{a}$ and $\frac{y}{b}$ are unequal, and if all the letters stand for positive numbers, then $\frac{lx+my}{la+mb}$ lies between $\frac{x}{a}$ and $\frac{y}{b}$.**

Proof. Since $\frac{x}{a}$ and $\frac{y}{b}$ are unequal, one of these must be the greater. Let $\frac{x}{a} > \frac{y}{b}$ and denote $\frac{y}{b}$ by k, so that $\frac{x}{a} > k$;

$$\therefore\ y = kb \text{ and } x > ka;$$

and $\because$ l and m are positive,

$$\therefore\ my = kmb \text{ and } lx > kla; \qquad (\textit{Art. 121})$$

$$\therefore\ lx + my > k(la + mb);$$

and $\because$ $la + mb$ is positive,

$$\therefore\ \frac{lx+my}{la+mb} > k; \qquad \therefore\ \frac{lx+my}{la+mb} > \frac{y}{b}.$$

In a similar manner, by putting $\frac{x}{a} = k'$, it can be shown that

$$\frac{lx+my}{la+mb} < \frac{x}{a};$$

$$\therefore\ \frac{lx+my}{la+mb} \text{ has a value lying between } \frac{x}{a} \text{ and } \frac{y}{b}.$$

The particular case when $y = b$ and $l = m = 1$ is as follows: If $\frac{x}{a}$ is not equal to 1, then $\frac{x+b}{a+b}$ lies between $\frac{x}{a}$ and 1.

Thus, if x and a are unequal and positive, the result of adding any positive number b to the numerator and denominator of $\frac{x}{a}$ is a fraction $\frac{x+b}{a+b}$, which is nearer to unity than $\frac{x}{a}$.

DEF. A fraction is called a **proper** or an **improper** fraction according as the numerator is less or greater than the denominator.

It has just been shown that **a proper fraction is increased and an improper fraction is diminished by adding the same number to both numerator and denominator.**

Thus $\frac{2}{3} < \frac{3}{4} < \frac{5}{6}$; $\frac{3}{2} > \frac{4}{3} > \frac{5}{4}$.

Corollary. **Between any two given fractions $\frac{x}{a}$ and $\frac{y}{b}$ lie an indefinite number of different fractions.**

For it has been shown that for *all* positive values of l and m, $\frac{lx+my}{la+mb}$ lies between $\frac{x}{a}$ and $\frac{y}{b}$.

Ex. 1. *If the fractions* $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ *are unequal, and* l, m, n *stand for any positive numbers, then* $\frac{lx+my+nz}{la+mb+nc}$ *lies between the greatest and least of the given fractions.*

Of the given fractions, let $\frac{x}{a}$ be the greatest, and denote its value by k, then

$$\frac{x}{a}=k; \quad \therefore\ x=ka; \quad \therefore\ lx=kla.$$

Also $$\frac{y}{b}<k; \quad \therefore\ y<kb; \quad \therefore\ my<kmb,$$

and $$\frac{z}{c}<k; \quad \therefore\ z<kc; \quad \therefore\ nz<knc;$$

$$\therefore\ lx+my+nz<k(la+mb+nc);$$

$$\therefore\ \frac{lx+my+nz}{la+mb+nc}<\frac{x}{a}.$$

In a similar manner, it can be proved that $\frac{lx+my+nz}{la+mb+nc}$ is greater than the least of the fractions $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.

EXERCISE LVII.

1. If $\frac{x}{y}=\frac{3}{4}$, find the values of

(i) $\frac{x+y}{y-x}$; (ii) $\frac{4x-3y}{2x+y}$; (iii) $\frac{5x-3y}{8x+7y}$; (iv) $\frac{x^2+y^2}{x^2-y^2}$.

2. If $2x=3y$, find the values of

(i) $\frac{x}{y}$; (ii) $\frac{x}{x+y}$; (iii) $\frac{x+2y}{x-y}$; (iv) $\frac{2x^2+y^2}{2x^2-y^2}$.

3. If $\frac{a+b}{a-b}=\frac{7}{3}$, find the values of

(i) $\frac{a}{b}$; (ii) $\frac{a}{a-b}$; (iii) $\frac{2a+b}{3a-b}$; (iv) $\frac{a^2+b^2}{a^2-b^2}$.

4. If $\frac{3a-2b}{5a+7b}=\frac{4}{17}$, find the values of

$$\text{(i) } \frac{a}{b}; \quad \text{(ii) } \frac{a+b}{a-b}; \quad \text{(iii) } \frac{5a-7b}{3a+2b}; \quad \text{(iv) } \frac{ab}{a^2+b^2}.$$

5. If $\frac{a}{b}=\frac{3}{2}$ and $\frac{x}{y}=\frac{4}{3}$, find the values of

$$\text{(i) } \frac{2ax+by}{ax-by}; \quad \text{(ii) } \frac{abxy}{(ax+by)^2}.$$

6. If $\frac{a}{b}=\frac{x}{y}$, prove that

(i) $\frac{a}{x}=\frac{b}{y}$; (ii) $\frac{a+b}{b}=\frac{x+y}{y}$; (iii) $\frac{a-b}{b}=\frac{x-y}{y}$;

(iv) $\frac{a+2b}{a-2b}=\frac{x+2y}{x-2y}$; (v) $\frac{a^2+b^2}{a^2-b^2}=\frac{x^2+y^2}{x^2-y^2}$; (vi) $\frac{a^2+b^2}{ab}=\frac{x^2+y^2}{xy}$.

7. If $\frac{x}{a(b-c)}=\frac{y}{b(c-a)}=\frac{z}{c(a-b)}$, show that $x+y+z=0$.

8. If $\frac{ax}{b-c}=\frac{by}{c-a}=\frac{cz}{a-b}$, show that

$$\text{(i) } ax+by+cz=0; \quad \text{(ii) } a^2x+b^2y+c^2z.$$

9. By eliminating two letters in succession, show that if $x+y+z=0$ and $2x-3y+4z=0$, then $\frac{x}{7}=\frac{y}{-2}=\frac{z}{-5}$.

10. If, in addition to the two equations given in Ex. 9, it is also given that $ax+by+cz=0$, prove that if x, y, z are not zero, then

$$7a-2b-5c=0.$$

[Put $\frac{x}{7}=\frac{y}{-2}=\frac{z}{-5}=k$ and substitute for x, y, z in $ax+by+cz=0$.]

11. By the method of Ex. 8, Art. 151, solve the following:

(i) $2x=3y$, $3x+4y=5$; (ii) $3x+4y=0$, $4x-5y=1$;

(iii) $px=qy$, $ax+by+c=0$.

12. If $3x-y=2$ and $2x+y=13$, find the value of $\frac{x}{y}$ and complete the solution as in Ex. 11.

13. If $x=3y+2z$ and $3z=5x+2y$, find the value of $\frac{2x-y}{z}$.

14. If $x+y+z=0$ and $ax+by+cz=0$, find the value of $\frac{y-z}{y+z}$ in terms of a, b, c.

15. If a, b, x stand for any positive numbers, prove that

$$\frac{a+x}{b+x} > \text{ or } < \frac{a}{b},$$

according as $a <$ or $> b$.

CHAPTER XXI.

FRACTIONS IN MEASUREMENT.

153. A Fraction as a Measure. If AB denotes the unit of length, and if the straight line AB is divided into 4 equal parts AP, PQ, QR, RB, the length of the line AR which contains

FIG. 24.

3 of these parts is denoted by $\frac{3}{4}$ of AB. Thus the symbol $\frac{3}{4}$ indicates that part of the unit which is contained in AR, and may therefore be said to **measure** * the length of AR. (See Art. 78.)

* For this use of the word "measure" see *Encyclopédie des Sciences mathématiques* (1904), Tome 1, p. 51.

From this point of view, **the fraction $\frac{a}{b}$ is defined as the measure of a magnitude obtained by dividing the unit into b equal parts, and taking a of these parts.**

Starting with this definition and taking a length as the magnitude to be measured, the properties of fractions are obtained as in the following Articles.

154. Equality and Inequality. The fraction $\frac{a}{b}$ is said to be $>$, $=$ or $<\frac{x}{y}$, according as the length measured by $\frac{a}{b}>$, $=$ or $<$ the length measured by $\frac{x}{y}$.

From this definition of the equality or inequality of two fractions, the rule of Art. 136 is obtained as follows:

Divide the unit of length AB into by equal parts and let k be the length of one of these parts, so that $AB = kby$;

$$\therefore \ \frac{a}{b} \text{ of } AB = kay \text{ and } \frac{x}{y} \text{ of } AB = kbx;$$

$$\therefore \text{ by definition, } \frac{a}{b} >, \ = \text{ or } < \frac{x}{y},$$

according to as $kay >, \ = \text{ or } < kbx$, that is according as

$$ay >, \ = \text{ or } < bx.$$

This is the Rule of Art. 136.

Ex. *Defining a fraction as a measure, prove that*

$$\frac{a}{b} = \frac{ax}{bx}.$$

Divide the unit into bx equal parts, then $\frac{1}{b}$ of the unit contains x of these parts, and $\frac{a}{b}$ of the unit contains ax of the parts;

$$\therefore \text{ by definition, } \frac{a}{b} = \frac{ax}{bx}.$$

155. Addition and Subtraction. $\frac{a}{c} + \frac{b}{c}$ *is* **defined** *as the measure of a length which is the sum of the lengths of which* $\frac{a}{c}$ *and* $\frac{b}{c}$ *are the measures.*

Hence $\frac{a}{c} + \frac{b}{c}$ is the measure of a length obtained by dividing the unit into c equal parts and taking $(a+b)$ of these parts;

$$\therefore \ \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Subtraction is defined as the inverse of addition, so that

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

156. Multiplication and Division. 1. Defining $\left(\frac{a}{b}\right)x$ to

mean the same as $\frac{a}{b}+\frac{a}{b}+\frac{a}{b}+\dots$ to x terms, it follows that

$$\left(\frac{a}{b}\right)x=\frac{a+a+a+\dots\text{ to }x\text{ terms}}{b}=\frac{ax}{b}.$$

In particular $$\left(\frac{a}{b}\right)b=\frac{ab}{b}=a.$$

2. Defining division as the inverse of multiplication,

$$\left\{\left(\frac{a}{b}\right)\div x\right\}x=\frac{a}{b}.$$

Now $$\left(\frac{a}{bx}\right)x=\frac{ax}{bx}=\frac{a}{b};$$

$$\therefore\ \frac{a}{b}\div x=\frac{a}{bx}.$$

3. If the length denoted by $\left(\frac{a}{b}\text{ of }AB\right)$ is divided into y equal parts, the length of 1 part is $\left(\frac{a}{by}\text{ of }AB\right)$, and the length of x parts is $\left(\frac{ax}{by}\text{ of }AB\right)$. Now the length obtained by dividing $\left(\frac{a}{b}\text{ of }AB\right)$ into y equal parts and taking x parts is denoted by $\left(\frac{x}{y}\text{ of }\frac{a}{b}\text{ of }AB\right)$;

$$\therefore\ \frac{x}{y}\text{ of }\frac{a}{b}\text{ of }AB=\frac{ax}{by}\text{ of }AB.$$

The symbol $\frac{\mathbf{a}}{\mathbf{b}}\times\frac{\mathbf{x}}{\mathbf{y}}$ *is* **defined** *as the measure of the length* $\frac{\mathbf{a}}{\mathbf{b}}$ *of* $\frac{\mathbf{x}}{\mathbf{y}}$ *of* AB, *where* AB *stands for the unit of length.*

Hence $$\frac{a}{b}\times\frac{x}{y}=\frac{ax}{by}.$$

Division is defined as the inverse of multiplication, so that

$$\frac{a}{b}\div\frac{x}{y}=\frac{ay}{bx}.$$

157. Summary. Starting with the definition of a fraction *as a measure*:

(i) It has been shown that fractions are to be compared by the rule of Art. 136.

(ii) The equations previously taken to define addition, subtraction, multiplication and division in the case of fractions have been established.

(iii) The equation $\left(\frac{a}{b}\right)b=a$, previously taken to define $\frac{a}{b}$, has been proved.

158. Representation of Numbers of the Rational System by Points. We can now find points on the scale constructed in Art. 107 which *represent* fractions.

FIG. 25.

To find the point L which represents $\frac{3}{4}$, divide 01 into 4 equal parts and set off $0L$ along the line, to the right of 0, to contain 3 of these parts.

To find the point M which represents $-\frac{3}{2}$, divide 01 into 2 equal parts and set off $0M$ to the left of 0 equal in length to 3 such parts.

Points constructed as above represent the numbers of the Rational Scale (*see Art.* 107) in the following respects:

(1) For every number there is one point and one only.

(2) The points occur in the order in which the corresponding numbers stand on the scale. This follows from Art. 154.

It follows from Art. 152 that between any two points, however close, which represent rational numbers, an indefinite number of points can be found which also represent rational numbers.

It will be shown later that the points on a straight line which represent rational numbers, although everywhere indefinitely close, *do not fill the line.*

Ex. 1. *Goods which cost £x are sold for £y (y being greater than x). What is the gain per cent.*

The sum invested is £x and the profit is £$(y-x)$, thus

$$\text{the gain on £}x \quad \text{is £}(y-x);$$

$$\therefore \text{ the gain on £100 is £}(y-x)\times\frac{100}{x};$$

$$\therefore \text{ gain per cent.} = \frac{100(y-x)}{x}.$$

Ex. 2. *A and B are travelling along the same road at the rates of u and v miles an hour respectively, and at noon B is a miles in front of A.*

(i) *If $u > v$, when will A overtake B?*

(ii) *Consider a case in which $u < v$; for instance let $u=4$, $v=6$, $a=2$, and interpret the result.*

(iii) *When is the result meaningless? Consider also a case where $u-v$ is a 'small' number; for instance let $u-v=0{\cdot}01$ and $a=2$.*

(i) Let A overtake B in x hours reckoned from noon. Then in x hours A walks a miles more than B; also in x hours A walks ux miles and B walks vx miles;

$$\therefore\ ux - vx = a,\ \ \therefore\ x(u-v) = a,\ \ \therefore\ x = \frac{a}{u-v}.$$

(ii) If $u=4$, $v=6$, $a=2$, we have $u-v=(-2)$, and

$$x = \frac{2}{-2} = -1.$$

We interpret (-1) hour *after* noon as meaning 1 hour *before* noon, and the conclusion is that *B passed A at 11 a.m.*

(iii) When $u=v$, $u-v=0$, and the equation $x(u-v)=a$ becomes $x\times 0=a$. As we are unable to divide by zero, we are unable to find a value of x which will satisfy this equation unless $a=0$, and then x may have any value whatever.

This corresponds to the fact that if A and B walk at the same pace and B starts a miles ahead, they will *never* be together unless $a=0$, and then they will *always* be together.

Next, let $u-v=0{\cdot}01$ and $a=2$; then $x=\dfrac{2}{0{\cdot}01}=200$. Thus, if $a=2$ and $u-v$ is small, x is large, and *by making $u-v$ small enough, we can make x as large as we please.*

Ex. 3. *A can do a piece of work in a days; B can do the same in b days. In how many days can they do it working together?*

Let x stand for the number of days taken by A and B to finish the work, when they work together; then

in 1 day A does $\frac{1}{a}$ of the work,

in 1 day B does $\frac{1}{b}$ of the work,

in 1 day A and B together do $\frac{1}{x}$ of the work;

$$\therefore \frac{1}{x}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}; \quad \therefore x=\frac{ab}{a+b}.$$

EXERCISE LVIII.

1. A grocer mixed p lbs. of tea at a shillings a pound with q lbs. at b shillings a pound. How much a pound was the mixture worth?

2. A train goes 120 miles at the rate of x miles an hour, and then 80 miles at the rate of y miles an hour. Write down an expression for the whole time taken.

3. A man can buy x articles for a certain sum, the price being a pence each. How many would he get for the same sum if the price were $(a+b)$ pence each? Also, what would be the price of each if he could buy $(x+y)$ articles for the same sum?

4. If x articles are obtained for £1, how many would be obtained for £1 if the price of each article was increased by 1*s.*?

5. A candle x inches long burns at the rate of y inches an hour. How many hours has it been burning when its length is reduced to z inches?

6. A man gets x articles for a certain sum of money, the price of each article being y pence. How many would he get for the same sum if the price of each article was $(y+z)$ pence?

7. A man was hired on the condition that for each day he worked he should receive b shillings, and for each day he did not work he should forfeit c pence. How many shillings were due to him at the end of x days, on y of which he had not worked?

8. A man is paid £a per month, and spends on an average $4b$ shillings a day. (i) How much does he save in the year? (ii) How many years will it take him to pay off a debt of £c?

9. Brine is formed by dissolving p pounds of salt in q pounds of water; how many pounds of salt are there in q pounds of brine?

10. n fish weigh m lbs. between them. Of these b average c lbs. each. What is the average weight of the remainder?

11. A workman's wages are a pence per hour for b hours in the week of regular time, and c pence per hour for overtime. One week he earns d shillings. How many hours' overtime did he work?

12. If a man buys a apples at the rate of b pence a score, and finds c of them rotten, at what price must he sell the remainder per dozen so as to gain or lose nothing on the transaction?

13. If q things were bought for p shillings and are sold at a profit of r per cent., for how many pence will each thing be sold?

14. If I buy a apples at p pence a dozen, and b apples at q pence a dozen and sell them all at one penny each, how much profit do I make, and what is the profit per cent. on the outlay?

15. A dealer buys a tons of coal at b shillings per ton and sells it at c pence per cwt. (i) What is his total gain (in pounds)? (ii) What is his gain per cent.? (iii) If he neither gains nor loses, state the relation connecting b and c.

16. A man takes £a to France, spends £b on the journey, and then changes the rest into French money at 25 francs the £. How long will it last him, at c francs a week?

17. Three men, A, B, and C, working separately, can do a piece of work in a, b, and c days respectively. In how many days can they do it when they work together?

18. A can do a piece of work in x hours, B can do it in y hours. How long will A and B take to do it, working together? After they have worked together for z hours, what fraction of the work remains to be done?

19. A can do a piece of work in m hours, B can do the same in n hours. If A works for p hours how long will B take to finish the work?

CHAPTER XXII.

EQUATIONS AND PROBLEMS.

159. Exceptional cases in the Solution of a Simple Equation.

Every equation of the first degree in x can be expressed in the form

$$xb = a. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

This equation has one solution and one only $\left(\text{namely } \frac{a}{b}\right)$, except when b is zero. In this case the equation becomes

$$x \times 0 = a.$$

As we are unable to divide by zero, this equation has no solution, unless a is also zero, in which case x may have any value, and the number of solutions is unlimited.

The student will find that cases in which an algebraical expression becomes meaningless, for certain values of the letters involved, deserve close attention. In the present case, although the equation (α) has no solution when b is zero, it has a solution *however small b may be.*

Let a remain constant, and let b have in succession the values 0·1, 0·01, 0·001, etc. The corresponding values of x are $10a$, $100a$, $1000a$, etc.

Note that as b diminishes the numerical value of x increases, and by making b small enough, the value of x can be made numerically greater than any number we choose. This is sometimes put briefly as follows: *When $b = 0$, the value of x determined by the equation $xb = a$ is infinite.**

For a problem illustrating this article, see Art. 158, Ex. 2.

* NOTE. It must be remembered that the statement in italics is nothing more than a convenient abbreviation for the longer statement which precedes it, and the student is warned against loosely regarding "infinity" as a number.

160. Equations with Fractions. In the case of an equation containing fractions, these may be removed from the equation by multiplying each side by the L.C.M. of the denominators, as in Art. 71, Ex. 2. Here we consider cases in which the unknown x occurs in the denominator. The next example will show that special precautions are necessary in such cases.

Ex. 1. *Search for a solution of the equation*

$$\frac{3x+4}{x(x-2)}=\frac{3}{x}+\frac{5}{x-2}. \quad \text{.........................}(\alpha)$$

Assuming that the equation has a solution and that x stands for a number which satisfies the equation, we multiply each side by $x(x-2)$, which is the L.C.M. of the denominators;

$$\therefore\ 3x+4=3(x-2)+5x; \text{.....................}(\beta)$$

$$\therefore\ x=2. \text{..................................}(\gamma)$$

We have thus shown that *if a solution exists, the solution is 2.* Now, when $x=2$, the given equation becomes

$$\frac{10}{2\times 0}=\frac{3}{2}+\frac{5}{0}.$$

As we are unable (at present) to assign meanings to $\frac{5}{0}$ and $\frac{10}{2\times 0}$, we cannot say that 2 is a solution.

We might also have tested the result of the search by considering whether the steps from (α) to (γ) are reversible (*see Art.* 23). The steps from (β) to (γ) are reversible, but (α) can only be obtained from (β) by dividing each side by $x(x-2)$, which is zero (for $x=2$). The step from (α) to (β) is therefore not reversible, and we cannot say that 2 is a solution.

Ex. 2. *Solve* $\frac{4x+3}{2x-5}=\frac{2x+1}{x-2}$.

Multiplying each side by $(2x-5)(x-2)$, we have

$$(4x+3)(x-2)=(2x-5)(2x+1);$$

$$\therefore\ 4x^2-5x-6=4x^2-8x-5;$$

$$\therefore\ 3x=1;$$

$$\therefore\ x=\tfrac{1}{3}.$$

Now the value $\frac{1}{3}$ of x does not make $(2x-5)(x-2)$, which is the L.C.M. of the denominators, equal to zero; hence the process is reversible and $\frac{1}{3}$ is the solution.

Ex. 3. *Solve* $\dfrac{3x-5}{2x+4}=\dfrac{3x-1}{2x+5}+\dfrac{1}{x+2}$.

The given equation is

$$\frac{3x-5}{2(x+2)}=\frac{3x-1}{2x+5}+\frac{1}{x+2}.$$

Multiplying each side by $2(x+2)(2x+5)$,

$$(3x-5)(2x+5)=2(x+2)(3x-1)+2(2x+5),$$

whence we find that $x=-3\frac{4}{9}$. This value of x does not make $2(x+2)(2x+5)$ zero; the process is therefore reversible, and $-3\frac{4}{9}$ is the solution.

Ex. 4. *Solve the equation*

$$\frac{2x-3}{x-4}+\frac{5(x-4)}{x-3}=\frac{2x+1}{x-2}+\frac{5(x-2)}{x-1}.$$

The equation might be solved by multiplying each side by $(x-1)(x-2)(x-3)(x-4)$; this process would be laborious, and it is better to proceed as follows:—

First method. The given equation may be written

$$\frac{2x-3}{x-4}-\frac{2x+1}{x-2}=5\left\{\frac{x-2}{x-1}-\frac{x-4}{x-3}\right\}.$$

Simplifying each side separately, we have

$$\frac{(2x-3)(x-2)-(2x+1)(x-4)}{(x-4)(x-2)}=5\,.\,\frac{(x-2)(x-3)-(x-4)(x-1)}{(x-1)(x-3)};$$

$$\therefore\ \frac{(2x^2-7x+6)-(2x^2-7x-4)}{(x-4)(x-2)}=5\,.\,\frac{(x^2-5x+6)-(x^2-5x+4)}{(x-1)(x-3)};$$

$$\therefore\ \frac{10}{(x-4)(x-2)}=5\,.\,\frac{2}{(x-1)(x-3)}.$$

Dividing each side by 10, and multiplying each side by $(x-4)(x-2)(x-1)(x-3)$, we have

$$(x-1)(x-3)=(x-4)(x-2);$$

$$\therefore\ x^2-4x+3=x^2-6x+8;$$

$$\therefore\ 2x=5;\quad \therefore\ x=\tfrac{5}{2}=2\tfrac{1}{2}.$$

This value of x does not make the L.C.M. of the denominators zero; the process is therefore reversible, and $2\frac{1}{2}$ is the solution.

Second method. We have

$$\frac{2x-3}{x-4}=\frac{2(x-4)+5}{x-4}=\frac{2(x-4)}{x-4}+\frac{5}{x-4}=2+\frac{5}{x-4},$$

$$\text{and}\quad \frac{x-4}{x-3}=\frac{(x-3)-1}{x-3}=1-\frac{1}{x-3},$$

and in the same way

$$\frac{2x+1}{x-2}=2+\frac{5}{x-2}; \qquad \frac{x-2}{x-1}=1-\frac{1}{x-1}.$$

Substituting these values for the fractions in the given equation, we have

$$\left(2+\frac{5}{x-4}\right)+5\left(1-\frac{1}{x-3}\right)=\left(2+\frac{5}{x-2}\right)+5\left(1-\frac{1}{x-1}\right);$$

$$\therefore\ 7+\frac{5}{x-4}-\frac{5}{x-3}=7+\frac{5}{x-2}-\frac{5}{x-1}.$$

Subtracting 7 from each side and then dividing each side by 5

$$\frac{1}{x-4}-\frac{1}{x-3}=\frac{1}{x-2}-\frac{1}{x-1};$$

$$\therefore\ \frac{(x-3)-(x-4)}{(x-4)(x-3)}=\frac{(x-1)-(x-2)}{(x-2)(x-1)};$$

$$\therefore\ \frac{1}{(x-4)(x-3)}=\frac{1}{(x-2)(x-1)}.$$

Multiplying each side by $(x-4)(x-3)(x-2)(x-1)$,

$$(x-2)(x-1)=(x-4)(x-3);$$

$$\therefore\ x^2-3x+2=x^2-7x+12;$$

$$\therefore\ 4x=10; \quad \therefore\ x=\tfrac{10}{4}=2\tfrac{1}{2}.$$

This value of x does not make the L.C.M. of the denominators zero; the process is therefore reversible, and $2\frac{1}{2}$ is the solution.

EXERCISE LIX.

Solve the following equations:

1. $\dfrac{1}{x}+\dfrac{1}{2x}+\dfrac{1}{3x}+\dfrac{1}{4x}=\dfrac{1}{6}.$ **2.** $\dfrac{0{\cdot}4}{x}=\dfrac{0{\cdot}5}{x}\left(1-\dfrac{0{\cdot}1}{x}\right).$ **3.** $\dfrac{0{\cdot}4}{x}=0{\cdot}25-\dfrac{0{\cdot}65}{x}.$

4. $\dfrac{x-3}{2x-3}=\dfrac{4x+5}{8x+1}.$

5. $\dfrac{10(x+1)}{2x+5}=\dfrac{5x-2}{x-2}.$

6. $\dfrac{2x+1}{3x+2}-\dfrac{4x+1}{6x+5}=0.$

7. $\dfrac{18x+20}{36}=\dfrac{8x-12}{10x-4}+\dfrac{x}{2}.$

8. $\dfrac{1}{2x-3}+\dfrac{x}{3x-2}=\dfrac{1}{3}.$

9. $\dfrac{3}{x-1}-\dfrac{2}{x-2}=\dfrac{1}{x-3}.$

10. $\dfrac{x-3}{2x+1}+\dfrac{2x-1}{4x-3}=1.$

11. $\dfrac{2x}{x-1}-\dfrac{x-1}{2x}=\dfrac{3}{2}.$

12. $\dfrac{5}{x}+\dfrac{2}{x-1}=\dfrac{3(2x-1)}{x(x-1)}.$

13. $\dfrac{3}{x}+\dfrac{4}{x-1}=\dfrac{5-x}{x(x-1)}.$

14. $\dfrac{2}{x-5}=\dfrac{1}{3}-\dfrac{x}{3x+4}.$

15. $\dfrac{4}{x-8}-\dfrac{2}{3x-24}+\dfrac{3}{2x-16}=\dfrac{29}{24}.$

16. $\dfrac{1+x}{2+3x}=\dfrac{1-8x}{1+12x}+1.$

17. $\dfrac{2x-1}{x-2}-\dfrac{x+2}{2x+1}=\dfrac{3}{2}.$

18. $\dfrac{x+6}{4(x-2)}=\dfrac{x+10}{6(x-2)}+\dfrac{1}{9}.$

19. $\dfrac{x-6}{4(x-2)}=\dfrac{x-10}{6(x-2)}+\dfrac{1}{9}.$

20. $\dfrac{x+6}{4(x-2)}=\dfrac{x+10}{6(x-2)}+\dfrac{1}{12}.$

21. $\dfrac{x}{x-2}-\dfrac{x}{x+2}=\dfrac{1}{x-2}-\dfrac{4}{x+2}.$

22. $\dfrac{(x+3)(x+4)}{(x-3)(x-4)}=\dfrac{x+9}{x-5}.$

23. $\dfrac{7}{x+2}-\dfrac{10}{x-5}+\dfrac{3}{x+4}=0.$

24. $\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}=\dfrac{x-7}{x+7}.$

25. $\dfrac{2x-\frac{1}{2}}{5}+\dfrac{x-3}{2x+\frac{1}{2}}=\dfrac{4x+1}{10}.$

26. $\dfrac{x+3}{x+1}-\dfrac{x+5}{x+7}=1-\dfrac{x^2-2}{x^2+8x+7}.$

27. $\dfrac{1}{x+1}-\dfrac{1}{x+2}=\dfrac{1}{x+3}-\dfrac{1}{x+4}.$

28. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+4}{x+5}-\dfrac{x+5}{x+6}.$

29. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}.$

30. $\dfrac{x-4}{x-3}-\dfrac{x-6}{x-5}=\dfrac{x-8}{x-7}-\dfrac{x-10}{x-9}.$

31. $\dfrac{x-5}{x-3}-\dfrac{x-4}{x-2}=\dfrac{x+2}{x+4}-\dfrac{x+3}{x+5}.$

32. $\dfrac{2x-11}{x-2}+\dfrac{x+4}{x-3}=\dfrac{x-5}{x+2}+\dfrac{2x+9}{x+1}.$

33. $\dfrac{2x+3}{3x-5}-\dfrac{9}{2x}=\dfrac{1}{2}+\dfrac{x-1}{6x-10}.$

34. $\dfrac{x}{x-1}-\dfrac{2x+1}{2x-1}-\dfrac{x-4}{x-3}+\dfrac{2x-9}{2x-7}=0.$

35. $\dfrac{2x+1}{x+3}-\dfrac{2x+3}{x+4}=4\left\{\dfrac{x+2}{2x-1}-\dfrac{x+1}{2x-3}\right\}.$

36. $\dfrac{3x+5}{4}+\dfrac{x+1}{x-7}-\dfrac{1}{2}=\dfrac{9x+7}{12}+\dfrac{2x-8}{2x-16}+\dfrac{1}{6}.$

Find x in terms of the other letters from the following equations:

37. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{2}{x}$.

38. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}=2$.

39. $\dfrac{a}{x-a}+\dfrac{x+a}{a}=\dfrac{x-b}{a}$.

40. $\dfrac{x+a}{a}+\dfrac{x}{x-a}=\dfrac{x-a}{a}$.

41. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$.

42. $\dfrac{2}{x-a}-\dfrac{1}{x-b}=\dfrac{x}{x^2-a^2}$.

43. $\dfrac{ax+b}{cx+d}+\dfrac{cx+d}{ax+b}=\dfrac{a^2+c^2}{ac}$.

44. $\dfrac{a}{bx-ac}-\dfrac{b}{ax-bc}=\dfrac{a^2-b^2}{abx+c^3}$.

45. $\dfrac{(a+b)(a+2c)}{(x-b)(x-c)}-\dfrac{a+b}{x-c}-\dfrac{c}{x-b}=0$.

161. Simultaneous Equations with Fractions. In the following, some of the denominators contain the unknowns.

Ex. 1. *Solve the equations*

$$\frac{7}{x}+\frac{3}{y}=21, \quad\ldots\ldots\ldots\ldots(\alpha)$$

$$\frac{10}{x}-\frac{4}{y}=1. \quad\ldots\ldots\ldots\ldots(\beta)$$

We eliminate y (or x) before attempting to get rid of fractions.

Multiplying each side of (α) by 4 and each side of (β) by 3,

$$\frac{28}{x}+\frac{12}{y}=84, \quad\ldots\ldots\ldots\ldots(\gamma)$$

$$\frac{30}{x}-\frac{12}{y}=3. \quad\ldots\ldots\ldots\ldots(\delta)$$

From (γ) and (δ), by addition,

$$\frac{58}{x}=87; \quad \therefore\ 58=87x; \quad \therefore\ x=\frac{58}{87}=\frac{2}{3}.$$

Substituting $\frac{2}{3}$ for x in (β),

$$\frac{10}{\frac{2}{3}}-\frac{4}{y}=1; \quad \therefore\ 15-\frac{4}{y}=1;$$

$$\therefore\ \frac{4}{y}=14; \quad \therefore\ 4=14y; \quad \therefore\ y=\frac{4}{14}=\frac{2}{7}.$$

Ex. 2. *Solve* $7x+\frac{3}{y}=21$; $10x-\frac{4}{y}=1$.

Here we find the values of x and $\frac{1}{y}$;

$$x=\frac{3}{2} \text{ and } \frac{1}{y}=\frac{7}{2}; \quad \therefore\ y=\frac{2}{7}.$$

162. Exceptional Cases in the Solution of Simultaneous Equations.

In Ex. 1–3 which follow, we shall consider the case of two simultaneous equations which, for certain values of the letters involved, become inconsistent (*see Art.* 62), and for other values of the letters become identical.

Ex. 1. *Find the values of x and y which satisfy the equations*

$$x-y=a, \quad \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$
$$x-(1+k)y=b. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

From (α), (β), by subtraction,

$$ky=a-b.$$

Dividing each side by k,

$$y=\frac{a-b}{k}.$$

Substituting for y in (α),

$$x-\frac{a-b}{k}=a; \quad \therefore\ x=a+\frac{a-b}{k}.$$

Ex. 2. *Consider the case when $k=0$ in* Ex. 1.

When $k=0$, the equations (α), (β) become

$$x-y=a, \quad x-y=b.$$

These equations are *inconsistent*, and have no solution unless $a=b$, in which case they are *identical* and the number of solutions is unlimited.

Ex. 3. *What is to be remarked about the solution of the equations in* Ex. 1, *if k stands for a 'small' number?*

Illustrate by giving the following special values to the letters:

(i) $k=0{\cdot}0001$, $a=2$, $b=1$; (ii) $k=0{\cdot}0001$, $a=1{\cdot}0003$, $b=1$.

(i) If $k=0{\cdot}0001$, $a=2$, $b=1$, we have $a-b=1$

and $$y=\frac{a-b}{k}=\frac{1}{0{\cdot}0001}=10000\,;\quad x=a+y=10002.$$

(ii) If $k=0{\cdot}0001$, $a=1{\cdot}0003$, $b=1$, we have $a-b=0{\cdot}0003$.

$$\therefore\ y=\frac{a-b}{k}=\frac{0{\cdot}0003}{0{\cdot}0001}=3\,;\quad x=a+y=3{\cdot}0003.$$

It will now be seen that if in the equations of Ex. 1, the numerical value of k is small compared with that of $(a-b)$, then the values of x and y which satisfy the equations are large, and that *by making k small enough we can make the values of x and y as large (numerically) as we please.*

A convenient abbreviation for the statement in italics is the following: *When $k=0$, the values of x and y are infinite.*

Thus the pair of **inconsistent** *equations*

$$x-y=a,\quad x-y=b$$

may be regarded as satisfied by a pair of **infinite** *values of x and y.*

EXERCISE LX.

Solve the following simultaneous equations:

1. $\dfrac{3}{x}+\dfrac{4}{y}=1,\quad \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{12}.$

2. $\dfrac{7}{x}+\dfrac{9}{y}=4,\quad \dfrac{1}{x}-\dfrac{1}{y}=4.$

3. $\dfrac{13}{x}-\dfrac{42}{y}=1,\quad \dfrac{22}{y}-\dfrac{27}{x}=2.$

4. $\dfrac{5}{7x}+\dfrac{1}{y}=1,\quad \dfrac{1}{y}-\dfrac{11}{3x}=-\dfrac{43}{3}.$

5. $\dfrac{5}{x}+2y=\dfrac{10}{3},\quad \dfrac{1}{x}-\dfrac{3}{2}y=\dfrac{2}{3}.$

6. $2x-\dfrac{7}{2y}=\dfrac{17}{3},\quad x+\dfrac{2}{3}=-\dfrac{7}{2y}.$

7. $\dfrac{1}{x}+\dfrac{1}{y}=9\left(\dfrac{1}{x}-\dfrac{1}{y}\right),\quad \dfrac{2}{x}+\dfrac{1}{y}=3\left(\dfrac{2}{x}-\dfrac{1}{y}\right)+1.$

8. $\dfrac{1}{7x}-\dfrac{y+1}{5}=1,\quad \dfrac{1}{5x}+\dfrac{y+1}{7}=\dfrac{7}{5}.$

9. $\dfrac{1}{3}\left(\dfrac{1}{x}-\dfrac{3}{y}\right)=\dfrac{1}{7}\left(\dfrac{3}{x}-\dfrac{1}{y}\right),\quad 3\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=8.$

10. $\dfrac{2+5x}{3x}=\dfrac{y+4}{2}=\dfrac{2xy+9x+2}{6x}.$ [*Divide numerator and denominator of the first and third fractions by x.*]

11. $x+2y+\frac{1}{z}=7,$

$3x+y-\frac{1}{z}=10,$

$x-y=2.$

12. $3x+\frac{4}{y}+\frac{5}{z}=-13,$

$4x+\frac{5}{y}+\frac{3}{z}=7,$

$5x+\frac{3}{y}+\frac{4}{z}=6.$

13. $\frac{5}{x}-\frac{3}{y}+\frac{1}{z}=a,$

$\frac{2}{x}+\frac{5}{y}-\frac{1}{z}=2a,$

$\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=a.$

163. Problems involving Fractions.

Ex. 1. *If the numerator of a certain fraction is doubled and the denominator is increased by 8, the resulting fraction is equal to $\frac{2}{3}$: if 1 is added to the numerator and to the denominator of the original fraction, the resulting fraction is equal to $\frac{3}{4}$. What is the original fraction?*

Denote the fraction by $\frac{x}{y}$: then

$$\frac{2x}{y+8}=\frac{2}{3}, \quad \text{.......} (\alpha)$$

$$\frac{x+1}{y+1}=\frac{3}{4}. \quad \text{.......} (\beta)$$

From (α) and (β) we have

$$3x-\ y=8, \quad \text{.......} (\gamma)$$

$$4x-3y=-1. \quad \text{.......} (\delta)$$

From (γ) and (δ) we find that $x=5$, $y=7$;

$$\therefore \text{ required fraction}=\frac{5}{7}.$$

Ex. 2. *If I walk to the station at the rate of 11 yds. in 5 seconds, I have 7 minutes to spare; if I walk at the rate of 13 yds. in 6 seconds, I am 3 minutes late. How far is it to the station?*

Let the distance to the station $=x$ yards;

when I walk 11 yds. in 5 seconds,

I walk 1 yd. in $\frac{5}{11}$ seconds,

I walk x yds. in $\frac{5x}{11}$ seconds.

In the same way, if I walk 13 yds. in 6 seconds,

I walk x yds. in $\frac{6x}{13}$ seconds.

Hence, if I walk at the second speed, it takes me $\left(\frac{6x}{13}-\frac{5x}{11}\right)$ seconds longer to walk to the station than if I walk at the first speed.

Now, by hypothesis, it takes me $(7+3)$ seconds longer to walk to the station in the second case than in the first;

$$\therefore \frac{6x}{13}-\frac{5x}{11}=10;$$

$$\therefore 66x-65x=10\times 13\times 11;$$

$$\therefore x=1430;$$

$$\therefore \text{distance to station}=1430 \text{ yds.}$$

Ex. 3. *A merchant reduces the price of an article by 2 per cent.; by how much per cent. must the sale be increased that there may be an increase of 3 per cent. in the gross receipts?*

Let the sale increase by x per cent.; then, if originally he sold N articles at P shillings each, since the sale increases x per cent., for every 100 articles originally sold, he now sells $(100+x)$ articles

„ N „ „ „ „ $\frac{100+x}{100}\,.\,N$ „

Since the price of each article is diminished by 2 %,

what sold for 100 shillings now sells for 98 shillings

„ „ P „ „ „ $\frac{98}{100}P$ „

In the second case he therefore sells $\frac{100+x}{100}\,.\,N$ articles at $\frac{98}{100}\,.\,P$ shillings each;

$$\therefore \text{gross receipts in second case}=\frac{100+x}{100}\,.\,N\,.\,\frac{98}{100}\,.\,P \text{ shillings.}$$

Now, originally, the gross receipts were NP shillings, and, by hypothesis, the gross receipts increase by 3 %;

$$\therefore \text{gross receipts in second case}=\frac{103}{100}NP \text{ shillings;}$$

$$\therefore \frac{100+x}{100}\,.\,N\,.\,\frac{98}{100}\,.\,P=NP\,.\,\frac{103}{100}. \quad \text{..........}(a)$$

Now NP is not zero; we can therefore divide each side of (α) by NP;

$$\therefore \frac{100+x}{100} \cdot \frac{98}{100} = \frac{103}{100};$$

$$\therefore 98(100+x) = 10300;$$

$$\therefore 98x = 500, \quad \therefore x = 5\tfrac{5}{49};$$

$$\therefore \text{ the increase in the sale} = 5\tfrac{5}{49}\,\%.$$

EXERCISE LXI.

1. The sum of three fractions is 2: the first exceeds the second by $\frac{1}{2}$; the second exceeds the third by $\frac{1}{4}$: find them.

2. A debt which might have been paid exactly with $5x$ half-sovereigns and x half-crowns was paid out of a ten-pound note, and the change was found to be equal to $15x$ half-crowns and x half-sovereigns. Find x and the amount of the debt.

3. A man performed a journey of 7 miles in $1\frac{1}{4}$ hours. He walked part of the way at 4 miles an hour and rode the rest of the way at 10 miles an hour. How far did he walk?

4. A farmer bought a certain number of sheep for 40 guineas. He lost one-tenth of them and sold the rest for half-a-guinea a head more than he had given for them, thus gaining 5 guineas. Find the number of sheep bought.

5. A tradesman bought three kinds of tea. Half the total quantity bought cost him 1*s*. 6*d*. per lb., one-third of it 2*s*. per lb., and the remainder 3*s*. per lb. He mixed all the tea together and sold the mixture at 2*s*. 6*d*. per lb., thereby gaining £1. 15*s*. What was the total quantity bought?

6. A man bought some tea at 1*s*. 8*d*. per lb. and mixed it with twice as much tea at 1*s*. 6*d*. per lb. One-sixth of the total amount having been damaged, he sold the remainder at 1*s*. 11*d*. per lb., and gained 3*s*. on the original cost. How many lbs. of each kind were there?

7. A man starts at 2 p.m. to walk to a place 13 miles off. He walks at a uniform speed till 4 p.m., when he increases his speed by one mile an hour and reaches his destination at 5.30 p.m. At what speed did he walk during the first 2 hours?

8. A boy walks a certain distance at the rate of 10 yds. in 7 seconds and is 2 minutes too soon. If he had walked at the rate of 13 yds. in 10 seconds, he would have been 1 minute too late. What is the distance?

9. A train is 5 minutes late when it does a certain journey at $29\frac{1}{2}$ miles an hour, and is 2 minutes late when it travels at 30 miles an hour. How many miles is the journey?

10. A train usually does a journey of 70 miles at 30 miles an hour. One day it is stopped and delayed 12 minutes, but by doing the remainder of the journey at 40 miles an hour it arrives at the proper time. Where did the stoppage take place?

11. A cyclist allows himself $1\frac{1}{2}$ hours for a journey of 15 miles going at his usual pace; but, after going a certain distance at that pace, he meets with an accident and is compelled to walk the rest of the way at 3 miles an hour. He arrives at the end of his journey 35 minutes late. How far had he ridden when the accident happened?

12. A tourist rode during the early part of the day at 9 miles an hour, and then walked at 4 miles an hour. Altogether he walked half a mile further than he rode. If he had ridden at 6 miles an hour during the whole time occupied by his journey, he would have gone $2\frac{1}{2}$ miles further. Find the number of hours he was riding and walking.

13. A certain number is added to both numerator and denominator of the fraction $\frac{5}{9}$. The resulting fraction is $\frac{4}{5}$. What number was added?

14. A certain fraction becomes equal to $\frac{1}{3}$ if 1 is added to its numerator, and equal to $\frac{1}{4}$ if 1 is added to its denominator. Find the fraction. 4 -

15. The denominator of a certain fraction exceeds its numerator by 1. Two other fractions are formed, one of them by adding 9 to the denominator and the other by subtracting 6 from the numerator of the original fraction. If these two fractions are equal, find the original fraction.

16. If the numerator of a fraction is doubled and the denominator increased by 7, the fraction becomes equal to $\frac{2}{3}$; if the denominator is doubled and the numerator increased by 2, the fraction becomes equal to $\frac{3}{5}$. What is the fraction?

17. If 3 is subtracted from both numerator and denominator of a certain fraction, the resulting fraction is equal to $\frac{2}{3}$; if 3 is subtracted from the numerator and added to the denominator of the original fraction, the result is equal to $\frac{1}{3}$. Find the original fraction.

18. If 1 is subtracted from numerator and denominator of a certain fraction, the resulting fraction is equal to $\frac{1}{2}$; if 19 is subtracted from numerator and denominator of the original fraction, the result is equal to $\frac{1}{3}$. Find the fraction.

19. A cyclist sets out to ride from A to B, going at the rate of 7 miles an hour. Another sets out 40 minutes later to ride from B to A, going 9 miles an hour. They meet at a point 2 miles nearer A than B; find the distance from A to B.

20. A fruiterer sold a certain quantity of oranges for £6. 10*s.* If he had sold 2 more oranges for a shilling, the same quantity would only have realised £5. 17*s.* How many oranges did he sell?

21. A and B are running in opposite directions along a straight road and B's pace is four-fifths of A's pace. At a certain instant they face each other and are 200 yds. apart; in the course of the next 3 minutes they pass each other, and at the end of the 3 minutes they are 1069 yds. apart. Find A's pace in yds. per minute.

22. A floor, whose width is three-fifths of its length, is stained all round the edges to a width of 2 feet. The cost, at $1\frac{1}{2}d.$ a square foot, is 15*s.* 6*d.* What is the length of the room?

23. A dealer sells bicycles so as to make 25 per cent. profit. A rival dealer, who obtains the same bicycles £1 cheaper and sells them £1 cheaper, makes $27\frac{1}{2}$ per cent. profit. What price does the first dealer pay for the bicycles?

24. A man had two creditors A and B, his debt to A being double that to B. After paying A 4*s.* in the £ and B in full, he had £10 left. If his whole estate is just sufficient to pay both A and B 10*s.* in the £, find the value of his estate.

25. A man derives a yearly income of £174 from money, one-half of which is invested at 6 per cent., one-third at 4 per cent. and the remainder at 3 per cent. What is the total amount invested?

26. A tradesman marks his goods at a certain rate per cent. above cost price, and after allowing a discount of 10 per cent. on the marked price, he makes a profit of $12\frac{1}{2}$ per cent. At what price does he mark goods which cost him a sovereign?

27. Two farmers are in dispute about a field of 5 acres; if it is adjudged to *A*, he will have half as many more acres than *B*. If it is adjudged to *B*, *A* will have three-quarters as much land as *B*. How many acres has each irrespective of the 5 acre field?

28. Rowing at a uniform rate, a man can row 5 miles down stream in 30 minutes, and back again in 40 minutes. Find the rate of the stream.

29. Two sums are invested in 3 per cent. stock at 80 and in 4 per cent. stock at 90, respectively. The amounts of stock obtained are the same, but the incomes differ by £5. Find the sums.

30. A tradesman sends in a bill for £12, part of which is for labour and the other part for materials. If the charge for labour had been twice what it was, and the charge for materials one-third of what it was, the amount of the bill would still have been £12. What was the charge for labour and what for materials?

31. At a fair *A* bought two horses for £91. He sold one for 5 per cent. more and the other for 3 per cent. less than the buying price. On the whole, he loses 5*s.* by the transaction. What did each horse cost?

32. A dealer bought 50 bicycles at £10. 10*s.* each; he sold a certain number of them at a profit of 20 per cent., and the remainder at three-quarters of his first selling price. His net profit was £64. 1*s.* How many bicycles were sold at a profit of 20 per cent.?

33. A man has £1000 invested partly at $3\frac{1}{2}$ per cent. and partly at 5 per cent. His total income is £44. 17*s.* 3*d.* Find the amounts of his investments.

34. A man's salary was increased by £20, but at the same time the income tax rose from 1*s.* to 1*s.* 2*d.* in the £, the result being that the man's net income was only increased by £10. Find his original salary.

35. A manufacturer reduces the price of his goods by $2\frac{1}{2}$ per cent. What percentage of increase in the sales after the reduction will produce an increase of 1 per cent. in the gross receipts?

CHAPTER XXIII.

GRAPHS.

164. The Linear Equation (*continued*). If x and y are variables, any linear equation in x and y (see *Art.* 85) may be written

$$ax + by + c = 0, \qquad\qquad (\alpha)$$

where the values of a, b, c (which may be positive, zero or negative, integral or fractional) are supposed to be known. The equation (α) may be written

$$y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right),$$

and denoting $\left(-\frac{a}{b}\right)$ by m and $\left(-\frac{c}{b}\right)$ by p, it is seen that (α) may be written in the form

$$y = mx + p, \qquad\qquad (\beta)$$

where the values of m and p are known.

Any linear equation in x and y may therefore be written in either of the forms (α) or (β).

Ex. 1. *In the equation $4x + 5y = 1$,* (i) *give to x the values -1, $-0{\cdot}25$, 0, $0{\cdot}25$, $0{\cdot}5$, 1, in succession, and find the corresponding values of y. Plot the points which represent these pairs of values and verify that the points lie in a straight line.*

(ii) *Take any other point on the line whose coordinates can be represented by numbers, fractional or integral, and verify that these numbers satisfy the equation $4x + 5y = 1$.*

(i) Corresponding values of x and y are shown in the table:

x	-1	$-0{\cdot}25$	0	$0{\cdot}25$	$0{\cdot}5$	1
y	1	$0{\cdot}4$	$0{\cdot}2$	0	$-0{\cdot}2$	$-0{\cdot}6$

(ii) It is seen that the point $(-0\cdot5, 0\cdot6)$ is on the line, and when $x=-0\cdot5$ and $y=0\cdot6$, $4x+5y=4(-0\cdot5)+5(0\cdot6)=1$,

$\therefore$ these values of x and y satisfy $4x+5y=1$.

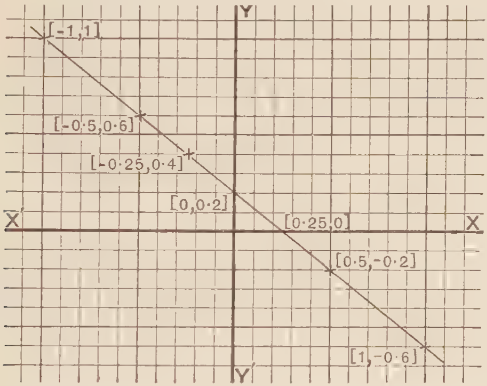

Fig. 26.

Experiments in plotting points whose coordinates satisfy a linear equation (as in Ex. 1) point to the fact that the conclusions of Art. 87 are true when x and y have fractional as well as integral values.

Ex. *Find graphically an approximate solution (correct to two places of decimals) of the equations*

$$234x+345y=678 \qquad \ldots\ldots(\alpha)$$

$$125x-157y=3 \qquad \ldots\ldots(\beta)$$

In (α), when $y=0$,

$$x=\tfrac{678}{234}=2\cdot90 \text{ (approx.)},$$

and when $x=0$,

$$y=\tfrac{678}{345}=1\cdot97 \text{ (approx.)}.$$

Taking 1″ as unit along OX and OY, plot the points A $(2\cdot90, 0)$, B $(0, 1\cdot97)$.

Join AB; then AB is the graph of (α).

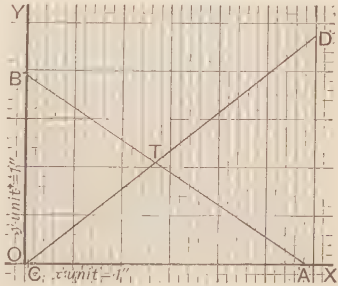

Fig. 27. (Half-Size.)

In (β), when $y=0$, $x=\frac{3}{125}=0{\cdot}02$ (approx.), and when $x=3$, $157y=125\times 3-3=124\times 3$, $\therefore\ y=\frac{372}{157}=2{\cdot}37$. Plot the points $C\,(0{\cdot}02,\ 0)$, $D\,(3,\ 2{\cdot}37)$. Join CD; then CD is the graph of (β).

These lines meet at T, whose coordinates are $(1{\cdot}36,\ 1{\cdot}05)$. The required solution is therefore $x=1{\cdot}36$, $y=1{\cdot}05$ (approx.).

In the last example observe that

(i) it is assumed that we can measure correctly to $0{\cdot}01$ of an inch. If we cannot do this, the figure must be drawn to a larger scale.

(ii) *In expressing the fractions as decimals, it is useless to carry the division beyond the second place of decimals unless we can measure to thousandths of an inch.*

If a straight line cuts the axes of x and y in A, B respectively, OA and OB are called the **intercepts** made by the line on the axes.

165. Gradient of a Line. From a point O (Fig. 28) draw a horizontal line OX and a straight line OY vertically upwards. Draw the straight lines OA and OB whose equations are $y=\frac{1}{5}x$ and $y=-\frac{1}{5}x$ respectively. Then in moving from O along OA, for any distance a feet traversed horizontally, there is a rise of $\frac{1}{5}a$ feet, and in moving from O along OB, for any distance a feet traversed horizontally, there is a fall of $\frac{1}{5}a$ feet. Taking "a *rise* of $(-\frac{1}{5}a)$ feet" to mean "a *fall* of $\frac{1}{5}a$ feet," it is seen that $\frac{1}{5}$ and $(-\frac{1}{5})$ measure the *rates at which a point rises* in proceeding from O along OA and OB. Hence $\frac{1}{5}$ and $(-\frac{1}{5})$ are called the *gradients* of OA and OB. Observe that the gradient of OA is *positive* and the line slopes *upward to the right*, whilst the gradient of OB is *negative*, and the slope is *downward*.

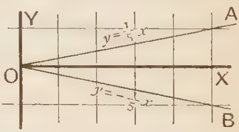

FIG. 28.

For the line whose equation is

$$y=mx+p, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

if x is changed to $(x+a)$, the corresponding value of y is $m(x+a)+p$, so that corresponding to an increment a in the value of x there is an increment ma in the value of y; thus

$$m=\frac{\text{increment in the value of } y}{\text{corresponding increment in the value of } x}$$

If m is negative, as x increases y decreases, and we must interpret *a negative increment as a decrement.*

The number m is called the **gradient** (or **slope**) of the straight line whose equation is $y=mx+p$; it measures *the rate of increase of* y and determines the angle which the line makes with OX.

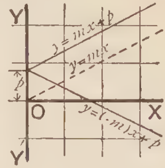

FIG. 29.

166. Lines through a given Point. For any value of m the equation

$$y-b=m(x-a)$$

represents a straight line through the point (a, b).

For the equation is linear and therefore represents a straight line; moreover the line passes through (a, b), for the equation is satisfied when $x=a$ and $y=b$.

167. Line through two given Points.

Ex. *Find the equation to the straight line which passes through the points* $(-3, 4)$, $(7, -5)$.

First Method. Let the equation to the line be $y=mx+p$.

The line passes through the points $(-3, 4)$, $(7, \;\; 5)$ if

$$4=m(-3)+p$$

$$\text{and} \quad (-5)=m\,.\,7+p.$$

Solving these equations for m and p, we find that $m=-\frac{9}{10}$ and $p=\frac{13}{10}$. The equation to the line is therefore

$$y=(-\tfrac{9}{10})x+\tfrac{13}{10} \text{ or } 9x+10y=13.$$

Second Method. The equation

$$y-4=m\{x-(-3)\} \text{ ....} (a)$$

represents a straight line through $(-3, 4)$.

This line passes through $(7, -5)$ if

$$(-5)-4=m\{7-(-3)\},$$

whence $m=-\frac{9}{10}$. Substituting $(-\frac{9}{10})$ for m in (a), the required equation is

$$y-4=-\tfrac{9}{10}(x+3),$$

which is the same as $9x+10y=13$.

168. Choice of Scales. It was stated in Art. 82 that it is not essential to the method of representing pairs of numbers by points that the same unit should be chosen for measurements along both axes. That the method may be *practically useful*, the units chosen should be as large as possible. In choosing the units, the size of the paper and *the largest numbers to be represented* must be considered.

Ex. *How is the equation $ax+by+c=0$ altered, by taking a new unit of measurement along OY, q times as great as the old unit?*

$$\text{Old unit for } OY = \frac{1}{q}(\text{new unit for } OY);$$

$$\therefore\ y(\text{old units}) = \frac{y}{q}(\text{new units}).$$

Hence the length originally denoted by y is now denoted by $\frac{y}{q}$, and the equation $ax+by+c=0$ becomes $ax+\frac{b}{q}y+c=0$.

This example shows that the *degree* of an equation is not altered by changing the units of measurement: hence a linear equation represents a straight line when the y-unit is different from the x-unit.

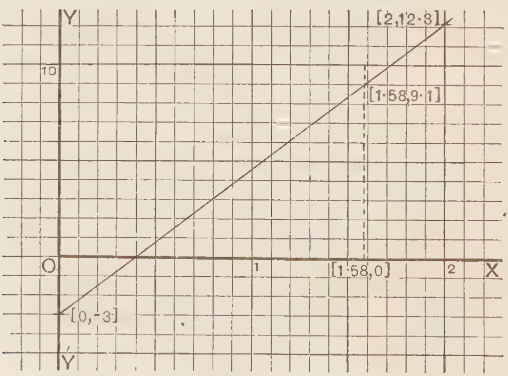

FIG. 30.

Ex. *Draw a graph of $\frac{23x-9}{3}$ for values of x from 0 to 2 and from 10 to 11. Read off the values of the expression for the values 1·58 and 10·34 of x.*

Let $y=\frac{23x-9}{3}$; when $x=0$, $y=-3$, and when $x=2$, $y=12{\cdot}3$ (nearly). Choose the units as described in Fig. 30. Plot the points $(0, -3)$, $(2, 12{\cdot}3)$; the line joining them is the graph of the expression. When $x=1{\cdot}58$, the graph shows that $y=9{\cdot}1$ (nearly).

It is convenient to show the part of the graph for values of x from 10 to 11 as in Fig. 31. When $x=10$, $y=73{\cdot}7$, and when $x=11$, $y=81{\cdot}3$; Fig. 31 shows parallels to the axes through (10, 70), and the point $(11, 81{\cdot}3)$ plotted in its proper relative position. The figure shows that when $x=10{\cdot}34$, the value of y is $76{\cdot}3$ (nearly).

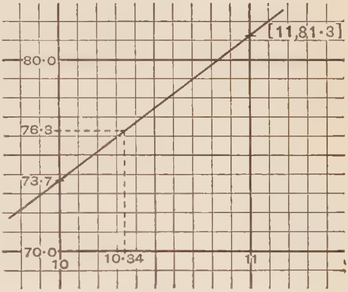

FIG. 31.

EXERCISE LXII.

1. What is the condition (or what are the conditions) that the straight line whose equation is

$$ax+by+c=0,$$

(i) passes through the origin? (iv) is parallel to the axis of y?
(ii) is parallel to the axis of x? (v) coincides with the axis of x?
(iii) passes through the point (f, g)?

2. (i) What is the gradient of the line whose equation is

$$ax+by+c=0?$$

(ii) What is the equation to a parallel to this line through the origin?
(iii) What is the equation to a parallel through the point (f, g)?

3. If the line whose equation is $ax+by+c=0$ cuts the axes of x and y in A, B respectively, what are the values of OA and OB?

4. (i) Prove by substitution that the line whose equation is

$$\frac{x}{a}+\frac{y}{b}=1$$

passes through the points $(a, 0)$, $(0, b)$.

(ii) What is the equation to the straight line which makes intercepts a and b on the axes?

5. On the axis of x take points A, A' each distant a units from O; on the axis of y take points B, B' each distant b units from O. Write down the equations to the straight lines AB, $A'B$, $A'B'$, AB'.

6. Let MP, NQ be the ordinates of points P, Q whose coordinates are $(1, m)$, $(m, -1)$. Prove that the triangles OPM, QON are congruent; hence show that the lines $y=mx$, $y=-\frac{1}{m}x$ are perpendicular. [Here the x-unit = the y-unit.]

7. By means of Ex. 6, show that the lines $y=mx+p$, $y=m'x+p'$ are perpendicular if $mm'=-1$. [Here the x-unit = the y-unit.]

8. Find the conditions that the straight lines whose equations are
$$ax+by+c=0, \quad a'x+b'y+c'=0$$
are (i) parallel; (ii) perpendicular.

9. Draw graphs of the straight lines referred to in (i)-(iv), and find the equation to each in the form $ax+by+c=0$.
   (i) The line through (3, 4) whose gradient is $\frac{2}{5}$.
   (ii) The line through $(-3, 4)$ whose gradient is $\frac{2}{5}$.
   (iii) The line through $(6, -5)$ whose gradient is $(-\frac{5}{2})$.
   (iv) The line through $(-6, -5)$ whose gradient is $(-\frac{5}{2})$.

10. Find the equation to the straight line joining the points in (i)-(vi), by both of the methods given in Art. 167.
    (i) $(1, 3)$, $(6, -7)$; (ii) $(6, -2)$, $(2, -8)$;
    (iii) $(-4, -2)$, $(3, 5)$; (iv) $(-1, -1)$, $(-5, -7)$;
    (v) $(0, 3)$, $(7, 8)$; (vi) $(0, -2)$, $(7, -8)$.

11. Prove that the three points in (i)-(iii) in each case lie on a straight line. Verify by plotting the points.
    (i) $(2, 1)$, $(7, 4)$, $(-3, -2)$; (ii) $(0, -3)$, $(9, 0)$, $(3, 2)$;
    (iii) $(3, 1)$, $(8, -1)$, $(-2, 3)$.

(To be proved by finding the equation to the line through two of the points and showing that the coordinates of the third point satisfy the equation.)

12. Find the equations to the straight lines through the point (3, 4) which are parallel to the lines whose equations are
    (i) $y=2x$; (ii) $x+y=1$; (iii) $3y-x+6=0$; (iv) $ax+by+c=0$.

13. Find the equations to the straight lines through the point $(1, -2)$ which are perpendicular to the lines in Ex. 12.

In Ex. 14–22, larger units than those indicated may be used with advantage, but the relative proportion of the x- and y-units should be as given.

14. Draw a graph of $375-(1{\cdot}23)x$ for values of x from 100 to 200. Read off the value of the expression, correct to the nearest integer, (i) when $x=134$; (ii) when $x=179$.
[Take x-unit $=$ y-unit $=0{\cdot}01''$.]

15. Draw a graph of $(38{\cdot}4)x-53{\cdot}7$ for values of x from 5 to 7. Read off the values of the expression, correct to the nearest integer, (i) when $x=5{\cdot}67$; (ii) when $x=6{\cdot}54$.
[Take x-unit $=0{\cdot}1''$, y-unit $=0{\cdot}01''$.]

16. Draw a graph of $(0{\cdot}17)x-45{\cdot}6$ for values of x from 200 to 300. Read off (i) the value of the expression, correct to 1 place of decimals, when $x=223$; (ii) the value of x when the expression is zero. [Take x-unit $=0{\cdot}01''$, y-unit $=0{\cdot}1''$.]

17. Draw a graph of $\frac{1}{9}(78x-87)$ for values of x from 0 to 10 and from 20 to 30. Read off the values of the expression to the nearest integer (i) when $x=7{\cdot}7$; (ii) when $x=25{\cdot}5$.
[Take x-unit $=0{\cdot}1''$, y-unit $=0{\cdot}01''$.]

18. Draw graphs of the expressions

$$\frac{23-37x}{5{\cdot}4} \text{ and } \frac{(3{\cdot}4)x-67}{0{\cdot}8}$$

for values of x from 0 to 20, and find, correct to the nearest integer, the value of each expression when their values are equal. [Take x-unit $=0{\cdot}1'$, y-unit $=0{\cdot}01''$.]

Obtain graphically the approximate solutions of the following simultaneous equations:

19. $\frac{x}{3{\cdot}34}+\frac{y}{2{\cdot}56}=1$; $23x=29y$. [Take x-unit $=$ y-unit $=1''$.]

20. $\frac{x+2}{58}=\frac{y+5}{75}$; $\frac{x}{18{\cdot}1}-\frac{y}{5{\cdot}2}=1$. [Take x-unit $=$ y-unit $=0{\cdot}1''$.]

21. $(11{\cdot}2)x-(1{\cdot}7)y=156$; $(3{\cdot}8)x+(0{\cdot}4)y=-9{\cdot}5$.
[Take x-unit $=0{\cdot}1''$, y-unit $=0{\cdot}01''$.]

22. $(0{\cdot}65)x-(0{\cdot}71)y=8{\cdot}5$; $(x+0{\cdot}45)(0{\cdot}71)=(0{\cdot}34-y)(0{\cdot}65)$.
[Take x-unit $=$ y-unit $=0{\cdot}1''$, and observe that the lines are at right angles.]

169. Applications.

Ex. 1. *Given that 1 kilometre = 0·62 mile (nearly), draw a graph showing the number of miles in any number of kilometres up to 100. Read off the number of kilometres in 43 miles to the nearest integer.*

Let y be the number of miles in x kilometres; then

$$1 \text{ km.} = 0{\cdot}62 \text{ mi.};$$
$$\therefore\ x \text{ km.} = (0{\cdot}62)x \text{ mi.};$$
$$\therefore\ y = (0{\cdot}62)x.$$

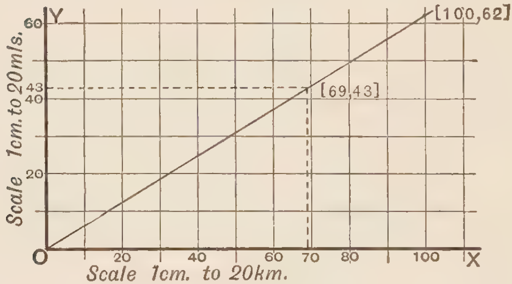

FIG. 32.

Choose the units as described in Fig. 32. Plot the point (100, 62); join this point to O. This line is the required graph.

The diagram shows that when $y = 43$, $x = 69$ (nearly). The number of kilometres in 43 miles is therefore 69 (nearly).

Ex. 2. *If a price of x shillings per lb. is the same as y francs per kilogram, exhibit graphically the relation between x and y. Read off the number of francs per km. corresponding to 33 shillings per lb. (Given that 20 shillings = 25 francs and 1 kilogram = 2·20 lbs.)*

$$\text{Cost of 1 lb.} = x \text{ shillings};$$
$$\therefore\ \text{cost of 1 kg. (or 2·20 lbs.)} = x \times (2{\cdot}20) \text{ shillings}$$
$$= x \times (2{\cdot}20) \times \tfrac{25}{20} \text{ francs}$$
$$= \tfrac{11}{4}x \text{ francs};$$
$$\therefore\ y = \tfrac{11}{4}x.$$

Choose the units as described in Fig. 33. Plot the point (36, 99); join this to O. This line is the required graph, which

shows that 33*s.* per lb. = 90 fr. per kg. (nearly).

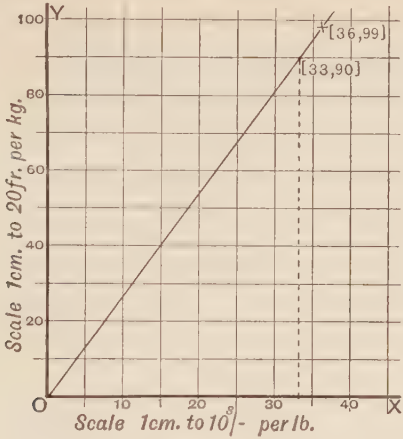

FIG. 33.

Ex. 3. *If a temperature of C degrees Centigrade is the same as one of F degrees Fahrenheit, exhibit graphically the relation between C and F.*

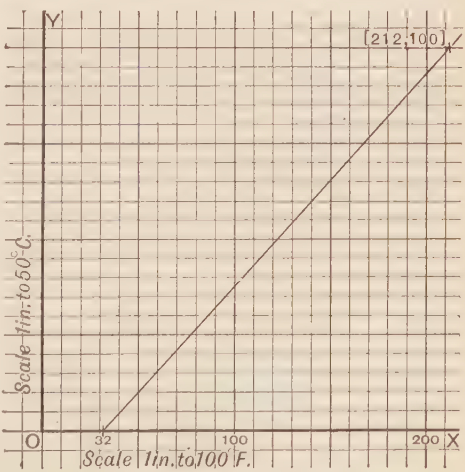

FIG. 34.

The freezing and boiling points of water are marked 0° and 100° Centigrade and 32° and 212° Fahrenheit; ∴ a temperature $F°$ Fahr. is $(F-32)$ degrees Fahr. above freezing, and a temperature of $C°$ Cent. is C degrees Cent. above freezing, thus

$(F-32)$ degrees Fahr. $=C$ degrees Cent.

and 180 degrees Fahr. $=100$ degrees Cent.;

$$\therefore \quad \frac{F-32}{180}=\frac{C}{100}; \qquad \therefore \quad F-32=\tfrac{9}{5}C.$$

This is a linear relation, and is represented graphically by a straight line. Take 1″ along OX to represent 100 degrees Fahrenheit and 2″ along OY to represent 100 degrees Centigrade. Plot the points given by $C=0$, $F=32$ and $C=100$, $F=212$. The required graph is the line joining these points.

The diagram in Fig. 34 is too small to be of much practical use. If we are concerned with a limited range of temperature, say from 50° F. to 60° F., we may proceed as follows:

By equation (α) we find that when $F=50$, $C=10$ and when $F=60$, $C=15{\cdot}6$ (nearly). We therefore represent the part of the graph in Fig. 34 from $F=50$ to $F=60$, on a magnified scale, as in Fig. 35, marking the origin as 50 on the scale of F and as 10 on the scale of C.

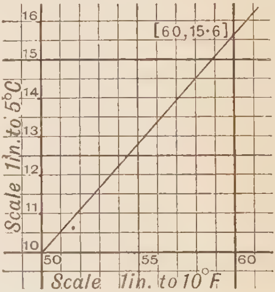

FIG. 35.

170. Two Physical Quantities connected by a Linear Equation.

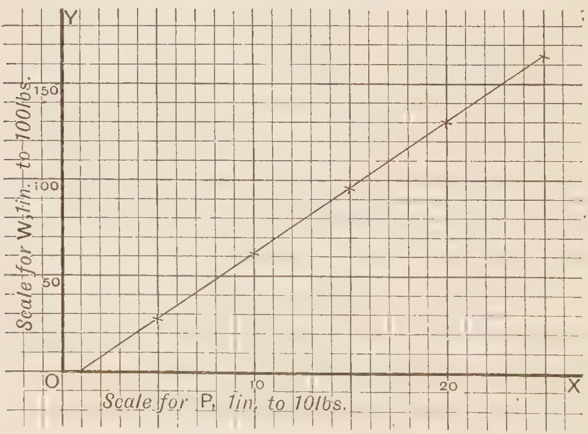

FIG. 36.

Ex. 1. *For a certain machine and for various loads, the table shows*

what effort of P lbs. was required to move a load of W lbs.

P	5	10	15	20	25
W	28	62	96	130	164

Plot points to represent these pairs of values of P and W. Show that we are justified in assuming a relation of the form $P = aW + b$, *where a, b are constants, and find the values of a and b.*

Choose the units as described in Fig. 36 and plot the points (5, 28), (10, 62), etc. The five points are found to lie in a straight line. *We are therefore justified in assuming* a relation of the form $P = aW + b$, where a, b are constants.

Substituting the values (5, 28) and (10, 62) in this equation, we have

$$5 = 28a + b, \text{ and } 10 = 62a + b.$$

Solving for a and b, we find $a = \frac{5}{34}$, $b = \frac{15}{17}$.

In the laboratory, the following problem often presents itself: A number of corresponding values of two mutually connected quantities, which we shall call x and y, are found by experiment. The problem is to find the equation which connects x and y. If the points which represent corresponding values of x and y are plotted, in many cases it will be found that these points lie approximately in a straight line. If in such a case we draw the straight line which passes *most evenly among the points and find its equation, we are justified in saying that this equation approximately represents the relation between x and y over the range of values of x dealt with in the experiments.*

It often happens that the points lie approximately in a straight line only when x lies between certain values h and k. We should then say that *if x lies between h and k, approximate values of y are given by an equation of the form* $ax + by + c = 0$, *where a, b, c are constants.*

Thus in Art. 90, Ex. 2, if the height of the bullet is y feet when its horizontal distance from O is x yards, since the part of the graph from $x = 0$ to $x = 200$ is nearly a straight line through O, we may say that between the limits 0 and 200 of x, approximate values of y are given by $y = kx$, where $k = \text{constant} \left(= \frac{14 \cdot 1}{200} \right)$. It would clearly be wrong to assume that this equation holds from $x = 200$ to $x = 1000$.

Ex. 2. *Approximate values of a quantity y, corresponding to certain values of another quantity x, are found by experiment to be as shown in the table:*

x	5	7	9	15	20	25	30
y	1·3	1·4	2·3	5·2	6·8	9·7	10·6

Plot points to represent the pairs of values of x and y. Assuming that x and y are connected by an equation of the form $y = mx + p$, find the approximate values of m and p.

Choose the units as described in Fig. 37 and plot the points;

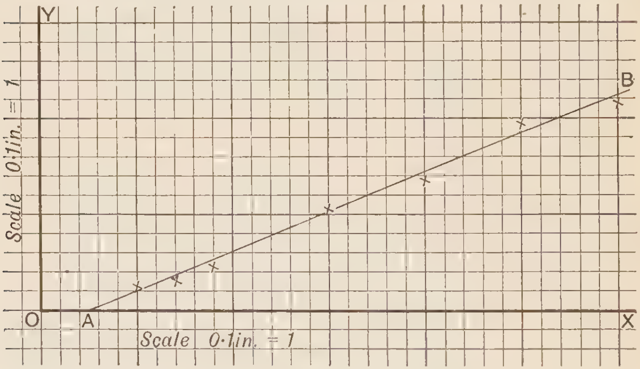

Fig. 37.

these are seen to lie approximately in a straight line. Draw the straight line AB, which passes as evenly as possible among the points. To find the equation to AB, choose two points A and B as far as possible apart on the line; let A be the point where the line cuts Ox and B the point whose abscissa is 30; find by measurement the lengths of OA and the ordinate of B. Thus at A, $x = 2·5$, $y = 0$, and at B, $x = 30$, $y = 11·1$.

Substituting these values in $y = mx + p$, we have

$$0 = (2·5)m + p,$$

$$11·1 = 30m + p;$$

whence $m = 0·404$, $p = 1·01$, and the equation connecting x and y may be taken as

$$y = (0·404)x - 1·01.$$

EXERCISE LXIII.

1. Given that 1 cub. foot contains 6·23 gallons, draw a graph to convert cubic feet into gallons. Scales, 10 cub. ft. to 1 cm. and 100 gallons to 2 cm. Read off the number of gallons in 93 cub. ft. and the number of cub. ft. in 615 gallons.

2. Given that 1 kilogram = 2·20 lbs., draw a graph to convert kilograms into pounds. Scales, 10 kg. to 1 cm. and 20 lbs. to 1 cm. Read off the number of pounds in 87 kg. and the number of kilograms in 87 lbs.

3. Draw graphs referred to the same axes to show the values of quantities of meat ranging from 10 to 20 lbs.,

(i) at $8\frac{1}{2}d.$ per lb.; (ii) at $11\frac{1}{2}d.$ per lb.

Scales, 2 cm. to 1 lb. and 1 cm. to $10d.$ Read off

(i) the value of $17\frac{3}{4}$ lbs. at $8\frac{1}{2}d.$;

(ii) value of $15\frac{1}{4}$ lbs. at $11\frac{1}{2}d.$;

(iii) amount of meat at $8\frac{1}{2}d.$ to be got for $10s.\ 2d.$ (correct to nearest $\frac{1}{4}$ lb.);

(iv) amount of meat at $11\frac{1}{2}d.$ to be got for $15s.\ 1d.$

4. Taking the value of a rupee at $1s.\ 4d.$, and assuming that an income of x rupees a month is the same as one of £y a year, what is the equation connecting x and y? Draw a graph to convert 'rupees a month' into '£ per annum' for incomes between 450 and 550 rupees a month. Read off the number of £ per annum equal to 514 rupees a month, and the number of rupees a month equal to £384 a year.

5. Draw a graph to express sums ranging from £20 to £30 as percentages of £80. Scales, 2 cm. to £1 and 1 cm. to 1 %. Read off, correct to 1 place of decimals, the percentages which £28. $17s.$ and £23. $11s.$ are of £80; also the values, correct to the nearest shilling, of 27·4 % of £80 and 36·8 % of £80.

6. Draw a graph to give the simple interest for 1 year, correct to the nearest penny, on sums ranging from £235 to £255. Scales, 1 cm. to a principal of £1, and 0·1 cm. to an interest of 1 penny. Read off the interest on £242. $9s.$ and on £251. $16s.$; also the sums which will produce an interest of £8. $10s.\ 4d.$ and of £8. $15s.\ 9d.$

7. A tradesman decides to mark his goods so that if he gives a discount of 1 penny in the shilling, he will still gain 35 % on the cost price of his goods. If an article which cost him x pence is marked y pence, what equation connects x and y? Draw a graph which would enable him to mark correctly to the nearest half-penny goods which cost him various sums up to 4*s.* 7*d.* Read off the price marked on articles which cost 2*s.* $5\frac{1}{2}$*d.* and 3*s.* 7*d.*; also the cost price of articles marked 1*s.* $11\frac{1}{2}$*d.* and 6*s.*

8. Draw a graph to convert 'miles per hour' into 'feet per second' for speeds up to 60 miles an hour. Read off speeds of 4 and 50 miles per hour as feet per second (to the nearest integer); also speeds of 22 and 59 feet per second as miles per hour.

9. The freezing and boiling points of water are marked 32° and 212° Fahrenheit and 0° and 80° Reaumur. Exhibit graphically the relation between these scales of temperature from the freezing to the boiling points. Read off the number of degrees Reaumur corresponding to temperatures of 60° F. and 180° F., and the number of degrees Fahrenheit corresponding to temperatures of 23° R. and 52° R.

10. In an examination, the top and bottom marks for a certain paper were 319 and 147. Draw a graph for reducing these so that the top counts 100 and the bottom 50. Read off the reduced marks corresponding to 183 and 275.

11. The total cost of providing a certain dinner is £20 when there are 100 guests, and £12 for 50 guests. If the total cost is C shillings when there are N guests, assuming that $C=a+bN$, where a and b are constants, (i) find the values of a and b.

(ii) Draw a graph showing the cost (to the nearest shilling) of the dinner corresponding to values of N from 50 to 100. Scales, 1 cm. to 5 guests and 1 cm. to 10 shillings; mark the origin 50 and 240 on the two scales.

(iii) Read off the cost for 93 guests and the number for whom dinner can be provided for £14. 10*s.* Verify your results by calculation from the formula obtained in (i).

12. An income tax of one shilling in the pound has of late been in force subject to the abatements given in the following table:

Exceeding.	Not exceeding.	Abatement.
...	£160	No tax.
£160	£400	No tax on first £160.
£400	£500	£150.
£500	£600	£120.
£600	£700	£70.
£700	...	No abatement.

Draw a graph to show the tax on all incomes from £150 to £900. If this tax were replaced by one of 1*s*. 8*d*. in the pound on the excess of each income over £300 (those not exceeding £300 being free of tax), between what limits must a man's income lie who would be benefited by the change?

13. For a certain machine and for various loads the table shows what effort P kilograms is required to move the load W kilograms:

W	50	100	200	300	400	500	600
P	106	122	152	184	216	246	278

Plot the values, and assuming the relation between P and W to be of the form $P = aW + b$, find the values of a and b.

14. A ball is thrown vertically upwards with a velocity of 160 feet per second. If after t seconds its vertical height above the point of projection is s feet, it can be shown that (neglecting the resistance of the air) $s = 160t - 16t^2$.

Tabulate the values of s for the values 0, 1, 2, 3, 4 ... 10 of t. Plot the corresponding values of t and s, and draw a smooth curve through the points so obtained. Obtain the correct shape of the curve for values from 4 to 6 of t by giving intermediate values to t. Take the axis of t horizontal and that of s vertically upwards; scale for t, 1 second to 2 cm.; scale for s, 20 ft. to 1 cm.

CHAPTER XXIV.

QUADRATIC EQUATIONS.

171. Zero Products. The following theorem is important: **If a product is zero, then one of the factors of the product is zero.**

Proof. (i) Let $ab=0$; then, if b is not zero, by the definition of division,

$$a=\frac{0}{b}=0.$$

(ii) Let $abc=0$; then, if neither a nor b is zero,

$$c(ab)=abc=0; \quad \therefore \; c=\frac{0}{ab}=0.$$

Proceeding in this way, the theorem can be proved for a product of any number of factors.

172. On Solving Equations. In the following examples the theorem of Art. 171 is applied to the solution of equations of a degree higher than the first.

Ex. 1. *For what values of x is the expression $x(x+2)$ equal to zero?*

The expression $x(x+2)$ is the product of the factors x and $x+2$, so that if $x(x+2)=0$, then *either* $x=0$ *or* $x+2=0$. Hence x may have either of the values 0 or -2.

Ex. 2. *Solve the equation $x^2=4$.*

Subtracting 4 from each side, we have

$$x^2-4=0; \quad \therefore \; (x-2)(x+2)=0.$$

$\therefore$ *either* $x-2=0$, giving $x=2$, *or* $x+2=0$, giving $x=-2$. The solutions of the given equation are therefore 2 and -2.

Ex. 3. *Find x in terms of a and b from the equation*

$$3(x-a+b)(x-a-b)=0.$$

If x has a value such that one of the factors of the left-hand

side of the equation is zero, then the equation is satisfied, and this value of x is a solution.

The only factors which can be zero are $x-a+b$ or $x-a-b$.

$\therefore$ *either* $x-a+b=0$, giving $x=a-b$;

or $x-a-b=0$, giving $x=a+b$.

The solutions are therefore $(a-b)$ and $(a+b)$.

The solutions of an equation containing a single unknown are called the **roots** of the equation; thus a and b are the roots of the equation $(x-a)(x-b)=0$.

Ex. 4. *Solve the equation* $(x+3)^2=4(x+3)$.

Subtracting $4(x+3)$ from each side, we have

$$(x+3)^2-4(x+3)=0;$$
$$\therefore\ (x+3)(x+3-4)=0;$$
$$\therefore\ (x+3)(x-1)=0.$$

$\therefore$ *either* $x+3=0$, giving $x=-3$, *or* $x-1=0$, giving $x=1$.

Ex. 5. *Criticise the following attempt to solve the equation*

$$(x+3)^2=4(x+3).$$

"*Divide each side of the equation by* $(x+3)$; $\therefore x+3=4$, $\therefore x=1$; *thus 1 appears to be the only solution of the equation.*"

Criticism. We cannot assign any meaning to the operation of dividing by zero. If then we divide each side of the equation by $(x+3)$, we assume that $(x+3)$ is not zero. Now when $(x+3)$ is zero, the given equation is satisfied and (-3) is also a solution.

173. Equivalent Equations. Two equations in an unknown x are said to be **equivalent** when every value of x which satisfies either of the equations also satisfies the other.

Theorem. *If by a reversible process an equation* (α) *in an unknown* x *is transformed into an equation* (β), *then the equations* (α) *and* (β) *are equivalent.*

Proof. By hypothesis, the equation (β) can be derived from (α); $\therefore$ every value of x which satisfies (α) also satisfies (β). Again, the process by which (β) is derived from (α) is reversible.

$\therefore$ every value of x which satisfies (β) also satisfies (α);

$\therefore$ the equations (α) and (β) are equivalent.

174. Irreversible Steps. We now give instances of processes which are not reversible for all values of x:

(1) Consider the equation

$$x=2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

Multiply each side by $(x-3)$;

$$\therefore\ x(x-3)=2(x-3). \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

The step which leads from (α) to (β) is not reversible for all values of x, for (α) is derived from (β) by dividing each side of (β) by $(x-3)$, and this is only possible on the assumption that $(x-3)$ is not zero.

We may state the matter thus: If $x=2$, it follows that $x(x-3)=2(x-3)$, but if it is given that $x(x-3)=2(x-3)$, then (as in Art. 172, Ex. 4) the conclusion is that x stands for *either* 2 *or* 3. The equations (α) and (β) are not equivalent, for (β) is satisfied by the value 3 of x which does not satisfy (α).

(2) If $x=2$ it follows that $x^2=4$, but if it is given that $x^2=4$, the conclusion is that x stands for *either* 2 *or* -2 (*Art.* 172, *Ex.* 2).

Thus the equations $x=2$ and $x^2=4$ are not equivalent, and *the process of squaring both sides of an equation is not reversible.*

175. Summary. The important facts to be gathered from the last three articles are these:

1. *In general, to multiply each side of an equation by an expression containing the unknown, or to square each side of an equation, is to introduce an extraneous solution (or extraneous solutions).**

2. *If each side of an equation is divided by a factor containing the unknown, in general a solution (or solutions) of the equation can be obtained by equating this factor to zero.*

176. Construction of an Equation whose Roots are given.

As an example, we will construct an equation whose roots are a, b, c.

The expression $(x-a)(x-b)(x-c)$ is zero if any one of its factors $x-a$, $x-b$, $x-c$ is zero, i.e., if x has any one of the values a, b, c.

* An extraneous solution is a solution which does not belong to the original equation.

Hence an equation whose roots are a, b, c is

$$(x-a)(x-b)(x-c)=0.$$

Moreover this equation has no other roots; for if the left-hand side is zero, one of its factors must be zero, so that x must have one of the values a, b or c. The equation

$$(x-a)(x-b)(x-c)=0$$

is therefore the 'simplest' equation whose roots are a, b, c.

It will be useful to obtain this equation in its expanded form: we have

$$\begin{aligned}(x-a)(x-b)(x-c) &= (x^2-ax-bx+ab)(x-c)\\ &= x^3-ax^2-bx^2+abx-cx^2+cax+bcx-abc\\ &= x^3-(a+b+c)x^2+(bc+ca+ab)x-abc.\end{aligned}$$

The equation is therefore

$$x^3-(a+b+c)x^2+(bc+ca+ab)x-abc=0.$$

Ex. 1. *Find an equation whose roots are 0, $(a+b)$ and $(a-b)$.*

Such an equation is

$$x\{x-(a+b)\}\{x-(a-b)\}=0.$$

Now
$$\begin{aligned}x\{x-(a+b)\}\{x-(a-b)\} &= x(x-a-b)(x-a+b)\\ &= x\{(x-a)-b\}\{(x-a)+b\}\\ &= x\{(x-a)^2-b^2\}\\ &= x(x^2-2ax+a^2-b^2).\end{aligned}$$

The equation in its expanded form is therefore

$$x^3-2ax^2+(a^2-b^2)x=0.$$

Ex. 2. *Obtain in its expanded form an expression which will be zero if $3x-2y=7$ or if $2x=5y$.*

Such an expression is $(3x-2y-7)(2x-5y)$;

now
$$\begin{aligned}(3x-2y-7)(2x-5y) &= 6x^2-4xy-14x-15xy+10y^2+35y\\ &= 6x^2-19xy+10y^2-14x+35y.\end{aligned}$$

An expression which satisfies the given conditions is therefore

$$6x^2-19xy+10y^2-14x+35y.$$

If an expression is zero for certain values of the letters which it contains, it is said to **vanish** for those values of the letters; thus the expression $6x^2-19xy+10y^2-14x+35y$ vanishes when $x=5$ and $y=2$, and when $x=7$ and $y=7$.

EXERCISE LXIV.

What are the roots of the following equations?

1. $x(x-1)=0$. **2.** $x^2-3x=0$. **3.** $x^2=4x$.

4. $(x-a)(x+b)=0$. **5.** $ax^2-bx=0$. **6.** $3x^2+2x=0$.

7. $9x^2=4$. **8.** $9x^2=-4x$. **9.** $(2x+1)(3x-2)=0$.

10. $x(x-1)(x+1)=0$. **11.** $x(x+a)(x+b)=0$.

12. $x(3x+4)(4x-5)=0$. **13.** $2x(x+1)=x(x+1)(x-3)$.

14. $(x-1)(x-2)(x-3)(x-4)=0$.

15. $(2x+1)(3x-1)(5x+2)(6x-5)=0$.

16. $x^3=a^2x$. **17.** $(x+2)x-3(x+2)=0$.

18. $(x-4)^2-2(x-4)=0$. **19.** $(x-1)^2=3(x-1)$.

20. $(2x+3)^2=(x-2)^2$. **21.** $9(x-1)^2-4(x+1)^2=0$.

22. $2x^2=50(x-3)^2$. **23.** $x^3+x^2-4x-4=0$.

24. $x^3-3x^2-x+3=0$. **25.** $2x^3-3x^2-18x+27=0$.

Find equations whose roots are the following, in each case giving the result in its expanded form:

26. $3, 2$. **27.** $-3, 2$. **28.** $3, -2$. **29.** $-2, -3$.

30. $0, a$. **31.** $-a, 0$. **32.** $3a, 5a$. **33.** $-3a, 5a$.

34. $3a, -5a$. **35.** $-3a, -5a$. **36.** $a, b+c$. **37.** $-a, b-c$.

38. $a+2b, a-2b$. **39.** $\frac{1}{3}, \frac{1}{5}$. **40.** $\frac{3}{4}, -\frac{1}{2}$.

41. $\frac{a}{c}, \frac{b}{c}$. **42.** $\frac{a+b}{c}, \frac{a-b}{c}$. **43.** $\frac{a+b}{a-b}, \frac{a-b}{a+b}$.

44. $a, b, -c$. **45.** $a, -b, -c$. **46.** $-a, -b, -c$.

47. $2, 3, 5$. **48.** $2, 3, -5$. **49.** $2, -3, -5$. **50.** $-2, -3, -5$.

Find expressions of the second degree in x and y which vanish in the following cases, in each case giving the result in its expanded form:

51. If $2x=3y$ or if $x=3$. **52.** If $x=y+6$ or if $2x=5y+1$.

53. If $3x-4y=1$ or if $x+y=10$. **54.** If $4x=7y+3$ or if $3x=5y-1$.

55. Prove that the expression

$$5x^2-16xy+12y^2-16x+20y+3$$

is zero (i) if $x=2y+3$, (ii) if $x=6t-1$ and $y=5t-1$, where t is any number whatever.

56. Prove that the expression

$$35x^2-68xy+32y^2-46x+44y+15$$

is zero (i) if $x=4s+3$ and $y=5s+3$, (ii) if $x=8t+3$, $y=7t+2$, where s and t stand for any numbers whatever.

177. Quadratic Equations. If a, b, c stand for any given numbers, the equation

$$ax^2+bx+c=0$$

is called a **quadratic** equation. It has been shown in Art. 171 that if a product is zero, then one of the factors of the product is zero; hence, if we can factorize ax^2+bx+c, we can solve the equation $ax^2+bx+c=0$.

Ex. 1. *Solve* $x^2+x-2=0$.

Since $x^2+x-2=(x-1)(x+2)$, we have

$$(x-1)(x+2)=0;$$

$$\therefore \textit{ either } \quad x-1=0, \text{ giving } x=1,$$

$$\textit{or } \quad x+2=0, \text{ giving } x=-2.$$

The solutions are therefore 1 and -2.

Ex. 2. *Solve* $9x^2+4=12x$.

The given equation is $9x^2-12x+4=0$;

$$\therefore (3x-2)^2=0.$$

Here the factor $(3x-2)$ occurs twice on the left-hand side, and we are only able to find one value of x which satisfies the equation, namely $\frac{2}{3}$. In such a case, instead of saying that the equation has only one root, we shall say that *it has two equal roots.*

A useful method is exhibited in the following example, which should be compared with that of Art. 42, Ex. 1:

Ex. 3. *Solve* $6x^2+17x+12=0$.

Multiplying each side by 6, we have

$$(6x)^2+17(6x)+72=0.$$

To factorize the left-hand side, we seek two numbers whose product is 72 and whose sum is 17; these are 8 and 9;

$$\therefore (6x+8)(6x+9)=0;$$

$$\therefore \textit{ either } \quad 6x+8=0, \text{ giving } x=-\frac{8}{6}=-\frac{4}{3},$$

$$\textit{or } \quad 6x+9=0, \text{ giving } x=-\frac{9}{6}=-\frac{3}{2}.$$

The solutions are therefore $-\frac{4}{3}$ and $-\frac{3}{2}$.

Ex. 4. *Solve* $10x=3\left(1+\frac{6}{x}\right)$. .... (α)

Multiplying each side by x, we have

$$10x^2=3(x+6);$$
$$\therefore\ 10x^2-3x-18=0;$$
$$\therefore\ (2x-3)(5x+6)=0; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

$\therefore$ *either* $2x-3=0$, giving $x=\frac{3}{2}$, *or* $5x+6=0$, giving $x=-\frac{6}{5}$.

Neither of these values of x is zero; the process which leads from equation (α) to equation (β) is therefore reversible when x has either of these values; therefore $\frac{3}{2}$ and $-\frac{6}{5}$ are solutions of (α).

Ex. 5. *Find x in terms of b and c from the equation*

$$x^2(b-c)+b^2(c-x)=0.$$

The equation can be written

$$(x^2b-xb^2)-(cx^2-cb^2)=0;$$
$$\therefore\ xb(x-b)-c(x^2-b^2)=0;$$
$$\therefore\ (x-b)\{xb-c(x+b)\}=0;$$
$$\therefore\ (x-b)\{x(b-c)-bc\}=0;$$
$$\therefore\ \textit{either}\quad x-b=0,\ \text{giving } x=b,$$
$$\textit{or}\quad x(b-c)-bc=0,\ \text{giving } x=\frac{bc}{b-c}.$$

The solutions are therefore b and $\frac{bc}{b-c}$.

EXERCISE LXV.

For what values of x do the following expressions vanish?

1. $2x^2-x$. **2.** x^2-3x+2. **3.** $x^2+4x-21$.

4. $x^2+11x+24$. **5.** $2x^2-x-1$. **6.** $2x^2-7x+3$.

7. $(5x-7)^2-9x^2$. **8.** $(3x+1)^2-16$.

9. $4(x+1)^2-(x-3)^2$. **10.** $5x^2-x-6$.

11. $2x^2-9x-18$. **12.** $18x^2-33ax+5a^2$.

Solve the following equations:

13. $(x-4)(x-5)=6$. **14.** $2(x-5)^2+3x=15$.

15. $(2x-3)(x+4)=9-6x$. **16.** $x^2+25=10x$.

17. $x^2-3x=88$. **18.** $x^2+11x=210$.

19. $(2x-1)(2x+1)=35.$

20. $(5+x)(5-x)=16.$

21. $8x^2=x(x+14).$

22. $(3x+2)(2x+3)=6.$

23. $6x^2+5=13x.$

24. $6x^2+7x=3.$

25. $12x^2+x=20.$

26. $36x(x-1)=7.$

27. $27x^2+12x=4.$

28. $5(2x^2-1)=7(x+1).$

29. $(3x+1)(8x-5)=1.$

30. $(2x+5)(8x+17)=1.$

31. $(3x-1)(2x+1)=3-x.$

32. $5(5x-4)(x-2)+9=0.$

33. $3x^2-7x=136.$

34. $2(x-7)^2+7x=109.$

35. $x-\dfrac{1}{x}=\dfrac{8}{3}.$

36. $x=\dfrac{5}{6}+\dfrac{1}{x}.$

37. $x-\dfrac{1}{10}=2+\dfrac{1}{x}.$

38. $9x=\dfrac{10}{x-1}.$

39. $\dfrac{15x}{15-x}=\dfrac{1}{x+1}.$

40. $2(x-1)=\dfrac{x-1}{x+1}.$

41. $2(x-3)+\dfrac{7}{x+2}=5.$

42. $x-\dfrac{x^3-8}{x^2+5}=2.$

43. $\dfrac{2x+1}{3}+\dfrac{3}{2x+1}=2.$

Find x in terms of the other letters from the following equations:

44. $(ax+b)^2=(bx-a)^2.$

45. $(ax+b)^2+(bx-a)^2=5(a^2+b^2).$

46. $(x+a)^3-(x-a)^3=2a(a^2+3b^2).$

47. $\dfrac{x-a}{2x-3a}=\dfrac{2a}{x}.$

48. $x^2-x(a-b)=ab.$

49. $ax+\dfrac{2b}{x}=2a+b.$

50. $ax^2-2x(ab+1)+4b=0.$

51. $x(x-b)=a(a-b).$

52. $x(x+2a)=b^2-a^2.$

53 $x(x-4b)=a^2-4b^2.$

54. $x(x+b)+a(a-b)=2ax.$

55. $x^2+(a+c)x=2a(a-c).$

56. $ab(x^2-1)=(a^2-b^2)x.$

57. $x^2(a+b)=a^2(x+b).$

178. Factors of ax^2+bx+c. Methods of factorizing ax^2+bx+c in special cases have already been given: a more general method consists in expressing ax^2+bx+c as the difference of two squares.

If $\left(\dfrac{b}{2}\right)$ is substituted for a in the identity

$$x^2+2ax+a^2=(x+a)^2,$$

the result is

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2.$$

Hence the result of adding $\left(\frac{b}{2}\right)^2$ to x^2+bx is a perfect square. Now $\frac{b}{2}$ is half the coefficient of x in x^2+bx; hence *if to x^2+bx we add the square of half the coefficient of x, namely $\left(\frac{b}{2}\right)^2$, the result is a perfect square.*

Ex. 1. *What must be added to the following expressions, so that in each case the result may be a perfect square for all values of x:*

$$\text{(i) } x^2+6x; \quad \text{(ii) } x^2-x; \quad \text{(iii) } x^2-\frac{1}{3}x?$$

The square of half the coefficient of x in (i) is 3^2 or 9, in (ii) is $\left(-\frac{1}{2}\right)^2$ or $\frac{1}{4}$, in (iii) is $\left(-\frac{1}{6}\right)^2$ or $\frac{1}{36}$. Thus the numbers to be added are (i) 9, (ii) $\frac{1}{4}$, (iii) $\frac{1}{36}$, and we have

$$x^2+6x+9=(x+3)^2,$$

$$x^2-x+\frac{1}{4}=\left(x-\frac{1}{2}\right)^2,$$

$$x^2-\frac{1}{3}x+\frac{1}{36}=\left(x-\frac{1}{6}\right)^2.$$

Ex. 2. *Express* (i) $x^2+2x-63$, (ii) $x^2-7x+12$ *as the difference of two squares; hence, factorize these expressions.*

(i)
$$\begin{aligned} x^2+2x-63 &= x^2+2x+1-64 \\ &= (x+1)^2-8^2 \\ &= (x+1+8)(x+1-8) \\ &= (x+9)(x-7). \end{aligned}$$

(ii)
$$\begin{aligned} x^2-7x+12 &= x^2-2\cdot\frac{7}{2}\cdot x+\left(\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2+12 \\ &= \left(x-\frac{7}{2}\right)^2-\left(\frac{49}{4}-12\right) \\ &= \left(x-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2 \\ &= \left(x-\frac{7}{2}+\frac{1}{2}\right)\left(x-\frac{7}{2}-\frac{1}{2}\right) \\ &= (x-3)(x-4). \end{aligned}$$

179. Solution of a Quadratic by completing the Square. *First method.*

Ex. 1. *Solve the equation $x^2+6x-40=0$.*

The square of half the coefficient of x is 9, and the equation may be written

$$x^2+6x+9-9-40=0;$$
$$\therefore\ (x+3)^2-49=0;$$
$$\therefore\ (x+3+7)(x+3-7)=0;$$
$$\therefore\ (x+10)(x-4)=0.$$

$\therefore$ *either* $x+10=0$, giving $x=-10$, *or* $x-4=0$, giving $x=4$. The solutions are therefore -10 and 4.

Ex. 2. *Solve the equation $6x^2-x-2=0$.*

The first step is to divide each side by 6, which is the coefficient of x^2, thus

$$x^2-\frac{1}{6}x-\frac{1}{3}=0.$$

The square of half the coefficient of x is $\left(-\frac{1}{12}\right)^2$ or $\frac{1}{144}$, and the equation may be written

$$x^2-\frac{1}{6}x+\frac{1}{144}-\frac{1}{144}-\frac{1}{3}=0;$$
$$\therefore\ \left(x-\frac{1}{12}\right)^2-\frac{49}{144}=0;$$
$$\therefore\ \left(x-\frac{1}{12}+\frac{7}{12}\right)\left(x-\frac{1}{12}-\frac{7}{12}\right)=0;$$
$$\therefore\ \left(x+\frac{1}{2}\right)\left(x-\frac{2}{3}\right)=0;\quad \therefore\ x=-\frac{1}{2}\ \text{or}\ \frac{2}{3}.$$

Second method. The following considerations lead to a method of solution which in practice is more convenient than the above.

The equation $x^2=a^2$ is equivalent to $x^2-a^2=0$, or to

$$(x-a)(x+a)=0,\ \text{giving}\ x=a\ \text{or}\ x=-a.$$

Hence if it is given that $x^2=a^2$, we may at once conclude that $x=a$ or $x=-a$. This conclusion is generally written shortly $x=\pm a$, where $\pm$ is an abbreviation for *plus or minus*.

Ex. 3.* *Solve the equation* $40 - x^2 = 6x$.

Arrange the equation so that the terms containing x are on the left and the coefficient of x^2 is $+1$, thus

$$x^2 + 6x = 40.$$

Add to each side the square of half the coefficient of x, namely 3^2;

$$\therefore\ x^2 + 6x + 9 = 49;$$
$$\therefore\ (x+3)^2 = 49;$$
$$\therefore\ x + 3 = \pm 7.$$
$$\therefore\ \textit{either}\quad x + 3 = 7,\quad \text{giving } x = 4,$$
$$\textit{or}\quad x + 3 = -7,\ \text{giving } x = -10.$$

The solutions are 4 and -10.

Ex. 4.* *Solve the equation* $6x^2 = x + 2$.

Arrange the equation so that the terms containing x are on the left, thus $6x^2 - x = 2$.

Divide each side by the coefficient of x^2;

$$\therefore\ x^2 - \frac{1}{6}x = \frac{1}{3}.$$

Add to each side the square of half the coefficient of x, namely $\left(-\frac{1}{12}\right)^2$ or $\frac{1}{144}$;

$$\therefore\ x^2 - \frac{1}{6}x + \frac{1}{144} = \frac{1}{3} + \frac{1}{144};$$
$$\therefore\ \left(x - \frac{1}{12}\right)^2 = \frac{49}{144};$$
$$\therefore\ x - \frac{1}{12} = \pm\frac{7}{12}.$$
$$\therefore\ \textit{either}\quad x - \frac{1}{12} = \frac{7}{12},\quad \text{giving } x = \frac{7}{12} + \frac{1}{12} = \frac{2}{3},$$
$$\textit{or}\quad x - \frac{1}{12} = -\frac{7}{12},\ \text{giving } x = -\frac{7}{12} + \frac{1}{12} = -\frac{1}{2}.$$

---

* The student should compare the working of the corresponding examples 1, 3 and 2, 4 line for line.

EXERCISE LXVI.

What must be added to the following expressions, so that in each case the result may be a perfect square, for all values of x?

1. x^2-8x.
2. x^2-5x.
3. x^2+x.
4. $x^2+\frac{2}{3}x$.
5. $x^2-\frac{1}{3}x$.
6. $x^2-\frac{6}{5}x$.
7. $x^2+\frac{b}{a}x$.
8. x^2+3ax.

Factorize the following by expressing each as the difference of two squares:

9. $x^2-12x+35$.
10. $x^2+14x+48$.
11. $x^2-10x-24$.
12. $x^2+2x-48$.
13. $x^2-9x-36$.
14. $x^2+11x-242$.
15. $2x^2+3x-2$.
16. $3x^2-x-10$.
17. $8x^2+10xy-3y^2$.
18. $10x^2+xy-11y^2$.
19. $x^2+(a+b)x+ab$.
20. $x^2-(a+b)x+ab$.

Find the values of x which satisfy the following equations by either of the methods of Art. 179:

21. $5x^2+2x=7$.
22. $14x=3(x^2+5)$.
23. $7x=4-2x^2$.
24. $8x^2=2x+3$.
25. $27x^2-2=3x$.
26. $8x=3(4x^2-5)$.
27. $14=18x^2+9x$.
28. $3x^2+20ax=32a^2$.
29. $ax+3a^2=10x^2$.
30. $2(x^2+7a^2)=11ax$.
31. $x^2-ax=b^2-\frac{a^2}{4}$.
32. $x^2-2ax=b(2a+b)$.
33. $(x-a)^2+2(x-a)=8$. (*Substitute y for $x-a$ and find the value of y.*)
34. $(x-2a)^2-4b(x-2a)=5b^2$.
35. $2\left(\frac{a}{x+a}\right)^2+5\left(\frac{a}{x+a}\right)-3=0$.

180. Equations with Fractions.

Ex. 1. *Solve* $\frac{1}{x+1}+\frac{2}{x+2}=\frac{6}{x+3}$.

Assuming that a solution exists and that x stands for a number which satisfies the equation, we multiply each side by

$$(x+1)(x+2)(x+3),$$

which is the L.C.M. of the denominators.

$$\therefore\ (x+2)(x+3)+2(x+1)(x+3)=6(x+1)(x+2);$$
$$\therefore\ x^2+5x+6+2(x^2+4x+3)=6(x^2+3x+2);$$
$$\therefore\ x^2+5x+6+2x^2+8x+6=6x^2+18x+12;$$
$$\therefore\ 3x^2+5x=0;\quad \therefore\ x=0 \text{ or } -\tfrac{5}{3}.$$

So far it has been shown that *the only values of x which can satisfy the given equation* are 0 and $(-\frac{5}{3})$. We can show by actual substitution, that these values do satisfy the equation: this also follows from the fact that *neither of these values of x makes* $(x+1)(x+2)(x+3)$ *zero*; $\therefore$ *the steps which lead from the given equation to* $x(3x+5)=0$ *are reversible*, $\therefore$ 0 and $-\frac{5}{3}$ are solutions.

Ex. 2. *Solve* $\dfrac{7}{x+3}+\dfrac{27}{x^2-9}=\dfrac{6x}{4x-12}.$

This equation may be written

$$\frac{7}{x+3}+\frac{27}{(x+3)(x-3)}=\frac{3x}{2(x-3)}.$$

Multiplying each side by $2(x+3)(x-3)$, we have

$$14(x-3)+54=3x(x+3);$$

$$\therefore\ 14x-42+54=3x^2+9x;$$

$$\therefore\quad 3x^2-5x-12=0;$$

$$\therefore\ (3x+4)(x-3)=0;\quad \therefore\ x=3 \text{ or } -\tfrac{4}{3}.$$

Now the value 3 of x makes $2(x+3)(x-3)$ zero, $\therefore$ when $x=3$ the steps which lead from the given equation to $(3x+4)(x-3)=0$ are not all reversible, and we cannot say that 3 is a solution. When $x=-\frac{4}{3}$, the expression $2(x+3)(x-3)$ is not zero; the above steps are all reversible and $-\frac{4}{3}$ is a solution.

181. Miscellaneous Examples on Quadratics.

Ex. 1. *Solve* $\left(\dfrac{5x-3}{3x+1}\right)^2-3\left(\dfrac{5x-3}{3x+1}\right)+2=0.$

The work can be shortened thus:—Substitute y for $\dfrac{5x-3}{3x+1}$ and the equation becomes

$$y^2-3y+2=0;$$

$$\therefore\ (y-1)(y-2)=0;\quad \therefore\ y=1 \text{ or } 2.$$

$$\therefore\ \textit{either}\quad \frac{5x-3}{3x+1}=1, \text{ giving } x=2,$$

$$\textit{or}\quad \frac{5x-3}{3x+1}=2, \text{ giving } x=-5.$$

Ex. 2. *Solve* $(5x-3)^2-3(5x-3)(3x+1)+2(3x+1)^2=0.$

Divide each side by $(3x+1)^2$ and the equation becomes that of Ex. 1.

Ex. 3. *Solve* $x^2+(0{\cdot}9)x=1{\cdot}12.$

Add to each side $(0{\cdot}45)^2$, which is the square of half the coefficient of x.

$$\therefore\ x^2+(0{\cdot}9)x+(0{\cdot}45)^2=1{\cdot}12+0{\cdot}2025;$$

$$\therefore\ (x+0{\cdot}45)^2=1{\cdot}3225.$$

By the usual arithmetical process, the square root of 1·3225 is found to be 1·15.

$$\therefore\ x+0{\cdot}45=\pm1{\cdot}15;$$

$$\therefore\ x=0{\cdot}7 \text{ or } -1{\cdot}6.$$

Ex. 4. *Solve* $3x^2+(0{\cdot}91)x=0{\cdot}049.$

To divide by 3 would introduce recurring decimals; we can proceed by expressing the decimals as vulgar fractions, or as follows:

Multiply each side by 3.

$$\therefore\ (3x)^2+(0{\cdot}91)(3x)=0{\cdot}147.$$

We now solve for $(3x)$ by adding to each side $(0{\cdot}455)^2$, which is the square of half the coefficient of $3x$.

$$\therefore\ (3x)^2+(0{\cdot}91)(3x)+(0{\cdot}455)^2=0{\cdot}147+0{\cdot}207025;$$

$$\therefore\ (3x+0{\cdot}455)^2=0{\cdot}354025;$$

$$\therefore\ 3x+0{\cdot}455=\pm0{\cdot}595;$$

$$\therefore\ 3x=0{\cdot}14 \text{ or } -1{\cdot}05;$$

$$\therefore\ x=0{\cdot}04\dot{6} \text{ or } -0{\cdot}35.$$

Ex. 5. *Solve* $a^2x^2+2a(b+1)x+b^2+2b-3=0.$

First method. Proceeding as in Art. 123, Ex. 1, we have

$$(ax)^2+2(b+1)(ax)+(b-1)(b+3)=0.$$

$$\therefore\ \{ax+(b-1)\}\{ax+(b+3)\}=0;$$

$$\therefore\ x=-\frac{b-1}{a} \text{ or } -\frac{b+3}{a}.$$

Second method. Solving for (ax), we have

$$(ax)^2+2(b+1)(ax)=-b^2-2b+3.$$

$$\therefore\ (ax)^2+2(b+1)(ax)+(b+1)^2=-b^2-2b+3+b^2+2b+1;$$

$$\therefore\ \{ax+(b+1)\}^2=4;$$

$$\therefore\ ax+b+1=\pm 2;$$

$$\therefore\ x=-\frac{b-1}{a}\ \text{or}\ -\frac{b+3}{a}.$$

EXERCISE LXVII.

MISCELLANEOUS EXERCISE ON QUADRATICS.

Solve the following equations:

1. $39x^2=2(8-x)$.

2. $91x^2+16x=35$.

3. $12x^2-37x=144$.

4. $154(x^2-1)=435x$.

5. $x^2-4x=6{\cdot}89$.

6. $x^2-(2{\cdot}4)x=4{\cdot}81$.

7. $x^2-(1{\cdot}3)x=0{\cdot}68$.

8. $(0{\cdot}4)x^2-(1{\cdot}08)x+0{\cdot}665=0$.

9. $\dfrac{x^2}{0{\cdot}3}+\dfrac{x}{0{\cdot}15}=0{\cdot}272$.

10. $3x^2+0{\cdot}896=(4{\cdot}16)x$.

11. $6x^2+0{\cdot}595=(4{\cdot}45)x$.

12. $7x^2+(1{\cdot}5)x=0{\cdot}0253$.

13. $\dfrac{3x+4}{4x+2}=\dfrac{6x+3}{7x+5}$.

14. $\dfrac{x}{x+4}+\dfrac{x}{4-x}=\dfrac{1}{x}$.

15. $\dfrac{2x+1}{3}=2-\dfrac{1}{x-6}$.

16. $\dfrac{x-1}{2x+1}-\dfrac{2x}{x-2}=2$.

17. $\left(x+\dfrac{1}{2}\right)^2-\dfrac{3}{2}(x+3)=\dfrac{13}{4}$.

18. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{13}{6}$.

19. $\dfrac{2}{x-3}-\dfrac{3}{x+1}=\dfrac{1}{2}$.

20. $\dfrac{1}{x-2}-\dfrac{1}{3-x}=\dfrac{3}{2}$.

21. $\dfrac{1}{x+1}-\dfrac{1}{2+x}=\dfrac{1}{x+10}$.

22. $\dfrac{1}{6(x+1)}+\dfrac{1}{3(x+2)}=\dfrac{1}{x+3}$.

23. $\dfrac{5(x+1)}{x+2}-\dfrac{5x}{x+1}=\dfrac{1}{6}$.

24. $\dfrac{4}{x^2-4}=\dfrac{1}{x-2}-\dfrac{x}{2(x+2)}$.

25. $\dfrac{2}{x-3}-\dfrac{1}{x-5}=\dfrac{1}{3x}+\dfrac{7}{3x(x-7)}$.

26. $\dfrac{1+2x-3x^2}{1-2x+3x^2}=\dfrac{3-2x+x^2}{3+2x-x^2}$.

27. $\dfrac{3(x-4)}{x-2}=\dfrac{2(5-x)}{6-x}-\dfrac{11}{5}$.

28. $\dfrac{1}{x-3}+\dfrac{4}{x(x+4)}=\dfrac{7}{60}+\dfrac{1}{x}$.

29. $\dfrac{17}{(2x-3)(3x+4)}=\dfrac{5x+1}{2x-3}-\dfrac{3}{3x+4}.$ **30.** $\dfrac{x-2}{x-3}=\dfrac{7}{12}-\dfrac{x}{4}-\dfrac{x^2}{4(2-x)}.$

31. $\dfrac{2(2x+3)}{x+1}=1+\dfrac{x^2-1}{2x^2+x-3}.$ **32.** $\dfrac{1}{2x+3}+\dfrac{1}{3x+2}=\dfrac{30(2x+3)}{(3x+2)^2}.$

33. $3(2x+1)^2+2(2x+1)(3x-1)=5(3x-1)^2.$

34. $20(2x+5)^2-11(2x+5)(11-4x)=3(11-4x)^2.$

35. $\dfrac{1}{3(3x-2)(5x-3)}-\dfrac{1}{2(5x-3)(2x-1)}=\dfrac{5x-3}{(2x-1)(3x-2)}-\dfrac{1}{6(5x-3)}.$

36. $\dfrac{2}{x}+\dfrac{1}{x-2}=\dfrac{2}{x+4}+\dfrac{1}{x-4}.$ **37.** $\dfrac{1}{x}+\dfrac{5}{3x-1}=\dfrac{1}{x+5}+\dfrac{10}{3(2x-1)}.$

38. $\dfrac{1}{x+1}-\dfrac{2}{x+5}=\dfrac{1}{2x+1}-\dfrac{3}{2(x+2)}.$

39. $\dfrac{4x+9}{x+1}-\dfrac{2x+15}{x+5}=\dfrac{4(x-1)}{x-2}-\dfrac{2(x+1)}{x-1}.$

40. $\dfrac{5x+4}{5x-6}+\dfrac{6}{13(2x+1)}=\dfrac{x+1}{x-1}+\dfrac{9}{13(3x-5)}.$

41. $\dfrac{5(2x+1)}{2x-1}+\dfrac{4(x-1)}{2x-3}=1+\dfrac{5(7x+4)}{7x-3}+\dfrac{3x-2}{3x-5}.$

Find the values of x which satisfy the following equations:

42. $\dfrac{5}{x+2a}+\dfrac{8}{x-a}=\dfrac{1}{a}.$ **43.** $\dfrac{3(a-b)}{x-b}-\dfrac{a}{x}=\dfrac{2(a+b)}{x+b}.$

44. $(2x-a)(2x-b)+a(x+4b)=(2x+a)(x+b).$

45. $(a-b)x^2-(c-a)x+b-c=0.$ **46.** $ab(x^2+ab)=x(a^3+b^3).$

47. $x(x+3)+a(a-3)=2(ax-1).$ **48.** $\left(\dfrac{a-x}{x-b}\right)^2-8\left(\dfrac{a-x}{x-b}\right)=15.$

49. $\dfrac{2a}{x-b}-\dfrac{b}{x-a}=1.$ **50.** $\dfrac{2}{x-a}-\dfrac{2}{x+c}=\dfrac{1}{a+c}.$

51. $a(a-1)x^2+(2a^2-1)x+a(a+1)=0.$

52. $(a^2-b^2)(x^2+1)=2(a^2+b^2)x.$ **53.** $(1-k)x^2+(1-k^2)x=\dfrac{2-k^2}{1-k}.$

54. $a(a-b)x^2-(a^2+ab+b^2)x+3b(a-b)=0.$

55. Find a value for x such that if x is substituted for a in $(a-3)(a-4)$, the value of this expression is unaltered.

56. What may be substituted for a in $\dfrac{(a-2)^2}{a-3}$ that the value of this expression is unaltered?

CHAPTER XXV.

PROBLEMS INVOLVING QUADRATICS.

182. Problems depending on Quadratics. The solution of each of the problems given below depends on a quadratic equation. In some of these it will be seen that *only one solution of the quadratic gives a solution of the problem:* the reason is that in every such case *the value of the unknown is limited by certain conditions stated or implied in the problem, and that these limitations do not enter into the algebraical statement contained in the equation.*

By the exercise of some ingenuity, it is often possible, as in Exx. 6 and 8 below, to state a kindred problem whose solution is given by the root of the quadratic which was inapplicable to the original problem. The examples just referred to may appear to be merely 'curious'; they nevertheless deserve the close attention of the advanced student, for the methods employed are of great use in the interpretation of apparently unintelligible results in many physical problems.

Ex. 1. *Find two positive integers whose sum is 15 and the difference of whose reciprocals is* $\frac{1}{18}$.

Let x and $15-x$ be the positive integers of which x is the less; then

$$\frac{1}{x}-\frac{1}{15-x}=\frac{1}{18}; \quad \text{.........................} (\alpha)$$

$$\therefore \frac{15-2x}{x(15-x)}=\frac{1}{18};$$

$$\therefore 18(15-2x)=x(15-x);$$

$$\therefore x^2-51x+18 \,.\, 15=0;$$

$$\therefore (x-6)(x-45)=0.$$

$\therefore$ *either* $x=6$, giving $15-x=9$, *or* $x=45$, giving $15-x=-30$.

In the problem, the value of x is limited by the conditions that both x and $(15-x)$ are to stand for positive integers. Hence the value 45 of x is inapplicable, and the required numbers are 6 and 9.

Ex. 2. *Find two numbers whose sum is 15 and the difference of whose reciprocals is* $\frac{1}{18}$.

The solution depends on equation (α) of Ex. 1; both values of x are applicable, and the answers are (6, 9) and (45, -30).

Ex. 3. *A man sold a horse for £27, and his loss per cent. was one-third of the number expressing the cost of the horse in pounds. What did the horse cost?*

Denote the cost price by £x;

$$\text{then loss per cent. on the horse} = \frac{x}{3};$$

$$\therefore \text{ actual loss} = \frac{x}{300} \text{ of } £x = £\frac{x^2}{300};$$

$$\therefore \text{ selling price} = £\left(x - \frac{x^2}{300}\right).$$

Now, by hypothesis, the selling price = £27.

$$\therefore x - \frac{x^2}{300} = 27; \quad \dots\dots\dots\dots(\alpha)$$

$$\therefore x^2 - 300x + 27 \times 300 = 0;$$

$$\therefore (x-30)(x-270) = 0.$$

$\therefore x = 30$ or 270, and both of these values of x satisfy the conditions of the problem; so that the information given in the problem only enables us to say that the cost price was *either* £30 *or* £270.

Ex. 4. *What is the price of eggs per dozen when a rise in price of 1 penny per dozen diminishes by 3 the number of eggs which can be bought for 5 shillings?*

If the price of eggs is x pence per dozen,

$$\text{for } x \text{ pence I can buy 12 eggs};$$

$$\therefore \text{ for 5 shillings I can buy } \frac{60}{x} \times 12 \text{ eggs or } \frac{720}{x} \text{ eggs.}$$

In the same way, if the price is $(x+1)$ pence per dozen,

$$\text{for 5 shillings I can buy } \frac{720}{x+1} \text{ eggs.}$$

By hypothesis, I obtain 3 fewer eggs in the latter case than in the former;

$$\therefore \frac{720}{x} - \frac{720}{x+1} = 3;$$

$$\therefore 720 = 3x(x+1);$$

$$\therefore x^2 + x - 240 = 0, \text{ giving } x = 15 \text{ or } -16.$$

The value -16 of x is inapplicable, so that the price was 15 pence a dozen.

Ex. 5. *Two pipes A and B, running together, can fill a cistern in 3 minutes, and, when running separately, B takes $1\frac{3}{4}$ minutes longer than A. How long does each pipe, running separately, take to fill the cistern?*

Let the times taken by A and B, running separately, to fill the cistern be respectively x and $(x+\frac{7}{4})$ minutes.

In 1 minute A fills $\frac{1}{x}$ of cistern,

" 1 " B " $\frac{1}{x+\frac{7}{4}}$ of cistern,

" 1 " A and B fill $\left(\frac{1}{x} + \frac{1}{x+\frac{7}{4}}\right)$ of cistern.

Now, by hypothesis, A and B can fill the cistern in 3 minutes;

$\therefore$ in 1 minute A and B fill $\frac{1}{3}$ of cistern.

$$\therefore \frac{1}{x} + \frac{1}{x+\frac{7}{4}} = \frac{1}{3}; \quad \ldots\ldots\ldots\ldots(\alpha)$$

$$\therefore \frac{1}{x} + \frac{4}{4x+7} = \frac{1}{3};$$

$$\therefore 3(8x+7) = x(4x+7);$$

$$\therefore 4x^2 - 17x - 21 = 0;$$

$$\therefore (4x-21)(x+1) = 0, \text{ giving } x = 5\tfrac{1}{4} \text{ or } -1.$$

The value (-1) of x is inapplicable; hence A and B take respectively $5\frac{1}{4}$ and 7 minutes to fill the cistern.

Ex. 6.* *State a problem of the same kind as that in* Ex. 5, *whose solution depends on the root* (-1) *of equation* (α).

To do this, we must *assign a meaning to every statement in the solution of* Ex. 5 *which is meaningless, as it now stands, when* $x = -1$. Observe that when $x = -1$, x is negative and $x + \frac{7}{4}$ is positive. We shall write $(-y)$ for x, so that equation (α) becomes

$$\frac{1}{\frac{7}{4} - y} - \frac{1}{y} = \frac{1}{3}. \qquad \text{.........................}(\beta)$$

Since (α) is satisfied by the values $5\frac{1}{4}$ and (-1) of x and $y = -x$, $\therefore$ (β) is satisfied by the values $(-5\frac{1}{4})$ and 1 of y.

Again, to say "in 1 minute A fills $\frac{1}{x}$ of the cistern" is to say "in 1 minute A increases the contents of the cistern by $\frac{1}{x}$ of the volume of the cistern." When x is negative and equal to $(-y)$, this must be altered to "in 1 minute A *decreases* the contents of the cistern by $\frac{1}{y}$ of the volume of the cistern." A would therefore *empty* the full cistern in y minutes.

Hence the root 1 of (β) is the solution of the following problem: *When two pipes A and B are both running, a cistern is filled in 3 minutes, and the sum of the times taken by A (running by itself) to empty the full cistern, and by B to fill the empty cistern, is $1\frac{3}{4}$ minutes. How long does B take to empty the full cistern?*

Ex. 7. *A and B are riding eastwards along a road at 10 and 16 miles an hour respectively, and B overtakes A at a certain point O on the road: C, who is travelling eastwards along the same road at x miles an hour, reaches O half an hour after A and B, and catches B an hour and a half after catching A. Find the value of x.*

When C reaches O, A is 5 miles ahead; also in 1 hour C gains $(x - 10)$ miles on A.

$$\therefore \text{ in } \left(\frac{5}{x-10}\right) \text{ hours } C \text{ gains 5 miles on } A;$$

$$\therefore\ C \text{ catches } A \left(\frac{5}{x-10}\right) \text{ hours after } C \text{ passes } O.$$

Similarly C catches B $\left(\frac{8}{x-16}\right)$ hours after C passes O.

* *May be omitted on first reading.*

Now, by hypothesis, C catches B $1\frac{1}{2}$ hours after catching A;

$$\therefore \frac{8}{x-16} - \frac{5}{x-10} = \frac{3}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$

From (α) we find that $x = 20$ or 8, and the value 8 of x is inapplicable, for if C travelled at 8 miles an hour, he would never catch A or B. Hence the value of x is 20, and C travels 20 miles an hour.

Ex. 8.* *Alter the problem in* Ex. 7 *so as to obtain a problem of the same kind whose solution is given by the root 8 of equation* (α).

To do this we must *assign a meaning to every step in the solution of* Ex. 7, *which is meaningless, as it now stands, when* $x = 8$.

Observe that when $x = 8$, both $(x - 16)$ and $(x - 10)$ are negative. Now to say "C catches A $\left(\frac{5}{x-10}\right)$ hours after C passes O," is to say "C and A are at the same point on the road $\left(\frac{5}{x-10}\right)$ hours after C passes O." Since $(x - 10)$ is negative, this must be altered to "C and A are at the same point on the road $\left(\frac{5}{10-x}\right)$ hours *before* C passes O; or seeing that A travels faster than C, we may say "A catches C $\left(\frac{5}{10-x}\right)$ hours *before* C passes O."

In the same way we must say "B catches C $\left(\frac{8}{16-x}\right)$ hours *before* C passes O."

Again, the equation (α) is the same as

$$\frac{8}{16-x} - \frac{5}{10-x} = -\frac{3}{2}.$$

If then (α) holds, it follows that C is caught by B $1\frac{1}{2}$ hours after C is caught by A.

The root 8 of (α) is therefore the solution of the problem obtained from that in Ex. 7 by altering the word "catches" into "is caught by" and the word "catching" into "being caught by."

---

* *May be omitted on first reading.*

EXERCISE LXVIII.

1. Find two consecutive integers whose product is 210.
2. Find two consecutive integers the sum of whose squares is 145.
3. Find two consecutive odd numbers the sum of whose squares is 290.
4. Find three consecutive integers the sum of whose squares is 365.
5. Find two consecutive numbers the difference of whose cubes is 217.
6. Find three consecutive odd numbers the sum of whose squares is 251.
7. Find the number whose square exceeds its double by 1443.
8. Find a number such that if its square is added to three times the number, the result is 154.
9. If 77 is added to three times the square of a certain number, the result is 40 times the number. Find the number.
10. Find two numbers whose sum is 50, such that the square of one of them is equal to five-ninths of the other.
11. Find two numbers whose sum is 10, such that their product added to the sum of their squares is equal to 76.
12. Find two numbers whose difference is 2 and the sum of whose squares is 74.
13. Twenty-eight years hence a man's age will be the square of what it was 28 years ago. Find his present age.
14. Find two consecutive odd numbers the sum of whose squares is 394.
15. The sum of two numbers is 50, and the sum of their reciprocals is $\frac{2}{21}$. Find them.
16. The difference of two numbers is 5, and the difference of their reciprocals is 0·01. Find them.
17. There are six consecutive odd numbers, and the sum of the products of pairs of these numbers taken equidistant from the first and last is 157. Find the numbers.
18. The area of a rectangle is 108 square feet; three squares are cut off from the rectangle by lines parallel to one of the shorter sides, and it is found that the length of the remainder of one of the longer sides is 10·5 feet. Find the lengths of the sides of the original rectangle.

19. A horse, bought for £x, is sold for £39 at a profit of x per cent. Find x.

20. A takes 5 days more to do a certain piece of work than B, and 9 days more than C; A and B together can do the work in the same time as C. How long would each take separately to do the work?

21. A sold an article which cost 50*s.* to B at a gain of x per cent.; B sold it back to A at a gain of x per cent. On the two transactions A lost 10*s.* 6*d.* Find x.

22. The circumference of the hind wheel of a wagon exceeds that of the front wheel by 1 foot, and the front wheel makes 22 more revolutions than the hind wheel in travelling a mile. Find the circumference of each wheel.

23. In a certain boat race the number of minutes occupied in the race was half the average number of strokes per minute and five times the number of miles rowed. The total number of strokes was 968. Find the length of the course and the time taken.

24. A number is formed of two digits which differ by unity. The sum of the squares of the number, and of the number formed by reversing the digits, is 1553. Find the number.

25. Each of one pair of opposite sides of a square field is shortened by 77 yds., and each of the other pair of sides is lengthened by 11 yds. If the area of the rectangular field thus obtained is 1·2 acres, find the side of the original square.

26. If I get two more eggs for a shilling when they are a penny a dozen cheaper, what is the price of eggs?

27. The price of oranges is such that if it were lowered by a half-penny a dozen, the number which could be bought for a shilling would be increased by 4. What is the present price per dozen?

28. A man bought a certain number of sheep for £30; after losing two of them he sold the rest for 10*s.* a head more than he had given for them, thereby gaining £2. 10*s.* How many sheep did he buy?

29. A man takes away £15 for his holiday expenses and finds that by reducing his expenses to the extent of 3 shillings a day, he can extend his holiday 5 days. How long was his extended holiday?

30. A bicyclist started 3 minutes later than he intended, to keep an appointment at a place 15 miles off. By going half a mile an hour faster than would have been necessary if he had started when he intended, he arrived at the proper time. At what rate did he ride?

31. A man having 26 miles to walk, starts at a certain pace, and after walking at that pace for an hour reduces his pace by a mile an hour. In consequence he takes an hour and a half longer on the journey than he would have taken if he had kept up his original pace. Find the pace at which he started.

32. A journey of 209 miles would be made by a train in 16 minutes less than the time actually taken if the speed were increased by 1 mile an hour. Find the speed of the train.

33. A draper buys 150 yds. of ribbon. Some of this is of fine quality, the rest of inferior quality and costs 1*s.* 6*d.* a yard less than the former. If he spends £6 on the dearer ribbon and £15 on the cheaper, what do the ribbons cost per yard?

34. A man went to market with £15 to lay out in oats. He found that he had to pay 2*s.* 6*d.* a sack more than he expected, and consequently came home with 10 sacks fewer than he expected. How many sacks did he buy and what did he pay for each?

35. The distances from A to B by two different routes are 77 miles and $97\frac{1}{2}$ miles. A motor, taking the longer road, travels 5 miles an hour faster than one taking the shorter road, and does the journey in 4 minutes less. Find the pace of each car.

36. A clock is 2 minutes slow, but is gaining. If it were 3 minutes slow, but were gaining half a minute more in a day than it does, it would show the correct time 24 hours sooner. How much does the clock gain in a day?

37. Two men A and B travel in opposite directions along a road 180 miles long, starting simultaneously from the ends of the road. A travels 6 miles a day more than B, and the number of miles travelled each day by B is equal to double the number of days before they meet. How many miles does each travel in a day?

TEST PAPERS.

1. (To § 7.)

1. Give an illustration of each of the three different ways in which letters are used to denote numbers. What is meant by the statement "$a=b$"?

2. If $x=1$, $y=2$, $z=3$, find the values of

(i) x^3+y^3; (ii) $y^2z^2+z^2x^2+x^2y^2$; (iii) $x^y+y^z+z^x$.

3. What is meant by (i) an identity, (ii) an equation? What value of x will make the equation $x+a=4a$ an identity?

4. Prove that $3\times 5=5\times 3$; also state (i) in words, (ii) algebraically the law of which this equality is an instance.

5. What is meant by saying that 3 is a solution of the equation $4x=12$? Prove that the values 2, 3 and 5 of x are solutions of the equation

$$x^3+31x=10x^2+30.$$

6. It takes x hours to row up stream from A to B at 3 miles an hour, and y hours to row down stream from B to A at 5 miles an hour.

(i) What is the equation connecting x and y?

(ii) If $y=3$, what is the value of x, and what is the distance from A to B?

7. Prove that $x^2+24=y^2+10x+2y$ in the following cases: (i) if $x=1$ and $y=3$; (ii) if $x=2$ and $y=2$; (iii) if $x=8$ and $y=2$; (iv) if $x=9$ and $y=3$.

8. Express as an equality the statement "x exceeds y by z." Show that, whatever numbers a, b, c stand for, a triangle can always be constructed whose sides contain respectively $(b+c)$, $(c+a)$, $(a+b)$ units of length.

2. (To § 17.)

1. Explain the meaning of a bracket. Describe in words the operations indicated in the following:

(i) $(a-b)\div(c-d)$; (ii) $a-b\div(c-d)$; (iii) $\{(a-b)-(c-d)\}^2$.

2. Find the value of each of the expressions in Ex. 1 when $a=9$, $b=3$, $c=8$, $d=5$.

3. Prove that $(4+5)\times 3=4\times 3+5\times 3$. Name the law of which this equality is an instance.

4. Factorize the following expressions:

(i) $x^3y^2+5y^2$; (ii) $(x+2y)^2+3x+6y$; (iii) $x(x+y)+xy+y^2$.

Test the correctness of each result by substituting 1 for x and 1 for y.

5. What is meant by "an algebraical expression"? How can you represent any odd number by means of an algebraical expression? Find the 250th odd number.

6. What is meant by saying that one number a is exactly divisible by another number b? When $x=5$, $y=3$, $z=1$, verify the following identities:

$$\text{(i)}\ (x^3-y^3)\div(x-y)=x^2+xy+y^2;$$
$$\text{(ii)}\ \{x^2-4(y-z)^2\}\div\{x-2(y-z)\}=x+2(y+z).$$

7. Express symbolically the cube of the remainder when the sum of b and c is subtracted from a, and find the value of the result when $a=7$ and $a+2(b+c)=17$.

8. (i) Find the value of $26\times 61+26\times 39+74\times 61+74\times 39$.

(ii) Find the sum of $a+2(3b+4c)$, $b+2(3c+4a)$, $c+2(3a+4b)$, and check the result by substituting 1 for a, 1 for b, 1 for c.

3. (To § 28.)

1. Simplify the expression $20x-3(x+y)-(5x-8y)$, writing out each step in full, and naming the law which justifies each step. Verify the result when $x=2$, $y=1$.

2. Find the sum of five consecutive odd numbers of which (i) x is the middle one; (ii) y is the greatest.

3. Find the remainder when $4(x^3+y^3)$ is subtracted from the sum of $4x^3-(5x^2y+3xy^2)$ and $6x^2y-(7xy^2-8y^3)$. What is the value of the remainder when $x=5$, $y=1$?

4. (i) State the converse of the theorem, "If $a=b$, then $a-x=b-x$." (ii) Prove that if $a<b$ and $x<y$, then $ax<by$. (iii) By considering the case when $a=12$, $b=10$, $x=6$, $y=1$, show that if $a>b$ and $x>y$, then *$a\div x$ may be less than $b\div y$*, and *$a-x$ may be less than $b-y$*.

5. (i) Solve the equation $7(x+1)=5(x+3)$. Name the law or quote the rule which justifies each step.

(ii) Find the value of a in order that the value 20 of x may satisfy the equation $2x+3a=5x-7a$.

6. A and B start at the same time from places 44 miles apart and cycle to meet one another, travelling at 10 and 12 miles an hour respectively. They meet x hours after starting. Find the value of x.

7. Prove the following identities :

(i) $(n+3)^2=3(n+2)^2-3(n+1)^2+n^2$;

(ii) $17(x+2)(x+3)+5(x+3)(x+4)-20(x+4)(x+2)=2(x^2+1)$.

8. If p, q, r, s are four consecutive numbers of which p is the least, use the first identity of Ex. 7 to express s^2 in terms of p^2, q^2 and r^2.

4. (To § 37.)

1. (i) Prove that $3.4.5=3.(4.5)$, and state in words the theorem of which this equality is an instance.

(ii) Assuming the commutative and associative laws for products of 3 factors, prove that $abcd=(ac)(bd)$.

2. State the three fundamental index laws. Multiply $6x^3y^2$ by $5x^4y^5$, writing out each step in full, and naming the law which justifies each step.

3. If $P=7a-2(b+c)$, $Q=7b-2(c+a)$, $R=7c-2(a+b)$, find in terms of a, b, c the values of (i) $P+Q+R$; (ii) $P+2Q-3R$.

4. Expand the following products :

(i) $(3x-2)(4x+3)$; (ii) $(5x+6)(6x+7)$; (iii) $(8x^2-5)(5x^2-1)$.

Check the results by substituting 1 for x.

5. Simplify

(i) $(2a+b)(4a^2-2ab+b^2)+(2a-b)(4a^2+2ab+b^2)$;

(ii) $5(a+b)(a-b)-(2a+b)(a-3b)-(3a+b)(a-2b)$;

(iii) $(a+b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2-(a-b)^3$.

6. Prove the following identities :

(i) $2(x-y)^2+2(3x-4y)^2=(2x-3y)^2+(4x-5y)^2$;

(ii) $(x+y+z)^2-(y+z-x)^2+(z+x-y)^2-(x+y-z)^2=8xz$;

(iii) $(x^3+3x^2y+3xy^2+y^3)(x^3-3x^2y+3xy^2-y^3)=(x^2-y^2)^3$;

(iv) $(x+a)(x+b)(x+c)-(x-a)(x-b)(x-c)=2x^2(a+b+c)+2abc$.

7. Find the H.C.F. and L.C.M. of $(2abc)^3$, $4a^2b^3c^6$, $2(a^2b^3c)^2$.

8. A has twice as much money as B, and when A has earned £450 and B has earned £100, A has 3 times as much as B. How much had each originally ?

5. (To § 41.)

1. If $x=2$, $y=3$, $z=1$, find the value of

(i) $\dfrac{(x+y+z)^3}{x^3+y^3+z^3}$; (ii) $x^{yz}+y^{zx}+z^{xy}$; (iii) $(yz)^x+(zx)^y+(xy)^z$.

2. Substitute $(y-1)$ for x in the expression $2x^3-3x^2+4x-5$, and arrange the result in descending powers of y.

3. Find the value of $(y+z-x)(z+x-y)(x+y-z)$ in terms of a, b, c, when $x=b+c$, $y=c+a$, $z=a+b$.

4. If $P=x^2+2xy+3y^2$, $Q=x^2-2xy-3y^2$, $R=x^2-2y^2$, find the value of R^2-PQ in terms of x, y. Verify the result when $x=4$ and $y=1$.

5. Simplify

(i) $(a+b)(x-y)+(a-b)(x+y)$; (ii) $(a+b)^3-(a^3+b^3)$;

(iii) $(2x+3)^4+(2x-3)^4-2(16x^4+81)$;

(iv) $(x-2y+3z)^2+6(x-2y+3z)(y-z)+9(y-z)^2$.

6. Add together ax^2-bx+c, bx^2-cx+a, cx^2+ax+b, bracketing the terms which contain the same powers of x; also express the sum as the product of two factors.

7. Factorize the following:

(i) $x^2-2xy-15y^2$; (ii) $x^4+2x^3-24x^2$;

(iii) $a^2-ac+bc-b^2$; (iv) $(2a+b)^2+6b(2a+b)+9b^2$;

(v) $(x+a)x^2-(x+a)(x+b)x+(x+a)ab$.

8. Given any three consecutive numbers, prove that the difference of the squares of the greatest and least is equal to four times the other number.

6. (To § 46.)

1. Express the sum of the squares of $ax+by$ and $bx-ay$ as the product of two factors.

2. Use the identity $a^2-b^2=(a+b)(a-b)$ to discover by how much the square of 69843 exceeds the square of 20157.

3. Find the value of $x^3-2x^2y+2xy^2-y^3$ in terms of a when $x=a+1$ and $y=a-1$.

4. Factorize the following:

(i) $x^2-7x-120$; (ii) $x^2-4xy-96y^2$; (iii) $6x^2+5xy-6y^2$;

(iv) $7x^2y-28y^3$; (v) $a^2-4b^2-12b-9$; (vi) $250-16x^3$.

5. What must be subtracted from $b(a^2+b-1)$ to produce $a(b^2+a-1)$ as a remainder? Express the result in factors.

6. Express as the product of as many factors as possible
(i) $(x^2-1)(x+2)+(x^2+2x)(x+1)$; (ii) x^7y+xy^7;
(iii) $(a+1)(10a^2+11a-6)+(3a-2)(2a^2+5a+3)$; (iv) $16x^6y^2-x^2y^6$.

7. If $x=a^3+a^2$, $y=a^2+a$, $z=a+1$, express as the product of factors (i) $x+y+z$; (ii) x^2-yz, and prove that

$$(x+y)(x+z)(x^2-yz)=(x+y+z)(x-z)(x^2+y^2).$$

8. A dealer buys $(a+b)$ tons of hay at £$(c-d)$ per ton and $(a-b)$ tons at £$(c+d)$ per ton, and sells the whole at £c per ton. What profit does he make?

7. (To § 50.)

1. From what expression must the sum of $5x^2-(3x+2)$ and $10x-3(x^2-5)$ be subtracted that the remainder may be $13x^2-5(3x+5)$? Express the result as the product of two factors.

2. Solve the following equations, and verify the solutions:

(i) $2(3x-2)-(2x-7)=9x+8-(6x-5)$;
(ii) $(5x+4)^2-(2x-3)(8x-5)=9(x+3)(x+5)$;
(iii) $x^3+(x-1)^3+(x-2)^3=3x(x-1)(x-2)+27$.

State a problem about three consecutive numbers whose solution depends on equation (iii).

3. Divide £80 into two parts so that one part is less than twice the other part by £19.

4. Prove that (i) the difference between the two numbers which can be formed with any two given digits is exactly divisible by 9;
(ii) the difference of their squares is exactly divisible by 99.

5. Factorize the following:
(i) x^3+x^2-42x; (ii) $4x^2-4xyz-15y^2z^2$; (iii) $x^2-2px-(q^2-p^2)$;
(iv) $(a+b)^2-3(a^2-b^2)-10(a-b)^2$; (v) $x^4-23x^2y^2+y^4$.

6. In a certain book, the solution of the following is asked for:

$$3(2x+\quad)(3x+2)-2(3x+1)^2=43,$$

a number in the first bracket being omitted by an oversight. The answer is given as $x=1$. Find the missing number.

7. If a stands for a natural number, find all the values of a for which $4x^2-ax-15$ can be resolved into factors.

8. Three men A, B, C share a legacy; C has £100 less than A, whilst A and C together have twice as much as B, and B and C together have £200 more than A. What was the legacy?

8. (To § 58.)

1. If m and n stand for two consecutive odd numbers, or for two consecutive even numbers, show that the difference of the cubes of m and n exceeds six times the product of m and n by 8.

2. Factorize the following :
(i) $24x^2-4x-48$; (ii) $1-4x^2-y^2-4xy$; (iii) $(2x-1)^2-(x-2)^2$; (iv) x^7y-xy^7 ; (v) $(x-3)(x-4)(x-5)+(x-1)(x-5)(x-9)$.

3. Solve the following equations, and verify the solutions :

(i) $3(x-5)(3x-4)-2(2x-9)(x+1)=5(x-4)^2-23$;

(ii) $(x+3)(x+4)(x-6)=(x-3)(x-4)(x+8)$;

(iii) $26x+21y=246$, $39x-14y=5$.

4. If $a>2b$, find x in terms of a, b, c from the equation

$$a(x-a-c)=2b(x-2b-c).$$

If a and b are connected by a certain relation, the above equation is satisfied for all values of x : what is this relation ?

5. If the expressions $12x-3y$, $17x-10y$ and $2x+y+50$ have equal values, find the value of x^2+y^2.

6. Prove that the sum of all the numbers which can be formed with the digits x, y, z, written in all possible orders, is $222(x+y+z)$. Verify this when the digits are 3, 5, 7.

7. Find x and y in terms of a and b from the equations

$$bx-ay=2x+y=2a+b.$$

8. A man buys $(4x+3a)$ lbs. of tea at 1*s.* 8*d.* per lb. and $(x+a)$ lbs. at 2*s.* per lb. and sells the whole at 1*s.* 9*d.* per lb.

What is his gain ? Show that the gain is the same for all values of a.

9. (To § 62.)

1. Factorize the following :

(i) $21a^2+4(ab+ac)-(b+c)^2$; (ii) $4(2x+3y)^2-9(x-y)^2$;
(iii) $(x-3)(x+1)^3+(x-1)^4-2(x-1)^3$; (iv) $27x^5y^2-8x^2y^5$.

2. Solve the following equations, and verify the solutions :

(i) $5(5x-6)-4(4x-5)=2x+16-3(3x-2)$;

(ii) $x+y-z=2x+y-4z=3(x-z)=6$.

3. (i) If $x=a-3$, find the value of $2x^3+11x^2+x-26$ in terms of a.

(ii) If $x^2=x+6$, prove that $x^5=55x+78$.

4. Prove that the following equations are inconsistent :

$$3x-4y+5z=6,\ \ 2x+3y-4z=8,\ \ 13x-6y+7z=41.$$

5. If $P=3m(x-1)^2-m(x-1)-4$, $\quad Q=16+n(x-1)-3n(x-1)^2$, find the value of $nP+mQ$.

6. Prove the following identity :

$$(x^2+2x+7)^2+(2x^2+2x-4)^2+(2x^2-6x+4)^2=(3x^2-2x+9)^2.$$

7. In the expression $ax^2+2bxy+cy^2$, substitute $X+Y$ for x and $X-Y$ for y; bracket the terms containing X^2, XY and Y^2. If the result is denoted by $AX^2+2BXY+CY^2$, write down the values of A, B, C in terms of a, b, c, and prove that

$$AC-B^2=4(ac-b^2).$$

8. A room a feet long and b feet wide is carpeted in such a way as to leave a margin c inches wide between the carpet and the walls.

Find the area of the margin in square inches.

10. (To § 73.)

1. Prove that the result of dividing a number a by two numbers b and c in succession is the same as the result of dividing a by the product of b and c, it being assumed that all the divisions can be performed.

2. Remove the brackets from the following :

$$\text{(i) } a\div(b\,.\,c\div d)\,; \quad \text{(ii) } a-(b+c-d).$$

Point out any similarity in the processes.

3. Simplify the following :

$$\text{(i) } \frac{25x^2y^4}{24a^4b^2}\div\frac{5x^3y}{12a^3b^5}\,; \quad \text{(ii) } \frac{(2a^4b^2-2a^2b^4)^2}{(2a^2b-2ab^2)^3}\,; \quad \text{(iii) } \frac{(x^2-x-30)(x^2-3x-28)}{x^2-2x-35}.$$

4. Solve the following equations, and verify the solutions :

$$\text{(i) } \frac{5}{7}(2x-11)-\frac{3}{4}(x-5)=\frac{x}{3}-(10-x).$$

$$\text{(ii) } x+y-\frac{1}{2}(x-y)=11\,; \quad x-y+\frac{1}{3}(x+y)=6\,;$$

$$\text{(iii) } \frac{x+a}{a+b}+\frac{x-3b}{a-b}=3, \text{ where } x \text{ denotes the unknown.}$$

5. Simplify the following, and verify the results by substituting 11 for x :

$$\text{(i) } \frac{2(x-2)}{3}-\frac{3(x-6)}{5}-\frac{x-5}{6}\,; \quad \text{(ii) } \frac{15(x-3)}{2x-2}-\frac{6(x-6)}{x-1}\,;$$

$$\text{(iii) } \frac{(2x-9)^2-(x-6)^2}{(x-5)^2}. \qquad \text{(iv) } \frac{3(x-8)^3(x-9)^2-(x-8)^2(x-9)^3}{2x^2-31x+120}.$$

6. If a and b are unequal and $ax-by=c(y-x)=c(a-b)$, find in terms of a and b the value of $(ay-bx)\div(a-b)$.

7. A cricketer finds that when he has played a matches his average is x runs, and that when he has played b matches more, his average is $(x+y)$ runs. What was his average for the last b matches?

8. A and B go for a holiday, A starting with £15 more than B. At the end of a week, A has spent a quarter of his money, B has spent a third of his, and together they have £58 left. How much money did each start with?

11. (To § 84.)

1. When is a number a said to be a perfect nth power? If a, b, c are perfect squares, prove that $\sqrt{a}\,.\sqrt{b}\,.\sqrt{c}=\sqrt{abc}$.

2. Explain the application of the identity

$$(a+1)^2=a^2+(2a+1),$$

as shown on the right-hand side, to the construction of a table of squares, and complete the table so as to show the squares of numbers from 980 to 985.

$$\begin{array}{r} 980^2=960400 \\ \underline{1961} \\ 981^2=962361 \\ \underline{1963} \\ 982^2=964324 \end{array}$$

3. Employ the results of Ex. 2 as follows:

(i) If $a=985$, $b=981$, find the value of $2ab$ by using the identity $(a-b)^2=a^2-2ab+b^2$.

(ii) By using the identity $(x+y)^2=x^2+2xy+y^2$, find the square of 985981.

4. If n stands for any number, prove that triangles the length of whose sides are denoted by the expressions in (i), (ii), (iii) are right-angled triangles.

(i) $n^2+1,\ n^2-1,\ 2n,\ (n>1)$;

(ii) $2n^2+2n+1,\ 2n^2+2n,\ 2n+1$;

(iii) $5n^2+14n+13,\ 3n^2+2n-5,\ 4n^2+16n+12,\ (n>1)$.

5. If $a=3$, $b=4$, $c=5$, $d=12$, find the values of

(i) $\sqrt{a^2+b^2+d^2}$; (ii) $\sqrt{(ac+bd)^2+(ad-bc)^2}$;

(iii) $\sqrt{(bd-ac)^2+(ad+bc)^2}$.

By means of (ii) and (iii) express the square of 65 as the sum of two squares in two different ways.

6. If I sell a horse for half as much again as I gave for it, I shall gain £20 more than I should lose if I sold the horse for three quarters of its original price. What did I give for the horse?

7. If $\frac{x+y}{3}=\frac{y+z}{4}=\frac{z+w}{8}$; $w=z+x$ and $z-y=2$, find the values of x, y, z, w.

8. A man has three sons A, B and C whose united ages just equal his own. Five years ago he was three times as old as A, two years ago he was three times as old as B, and one year ago he was four times as old as C. Find the present ages of father and sons.

12. (To § 91.)

1. The equations to the sides BC, CA, AB of a triangle are

$$2y-x=2, \quad x+y=16, \quad 2x-y=2.$$

Find graphically the coordinates of the vertices, and explain how to verify the results. If the unit of length is 0·1 in., calculate the area of the triangle in square inches.

2. Simplify the following :

(i) $\dfrac{a^2(b-c)-b^2(a-c)}{ab-c(a+b)}$; (ii) $\dfrac{(x+a)^4-13a^2(x+a)^2+36a^4}{x^2-3ax+2a^2}$;

(iii) $\dfrac{(3a+2b)^2-(2a+b)^2}{(7a-2b)-(2a-5b)}$; (iv) $\dfrac{(20x^2-13xy-21y^2)(6x^2-5xy-4y^2)}{10x^2-9xy-7y^2}$.

3. If $y+z=2a$, $z+x=2b$, $x+y=2c$, find x, y, z in terms of a, b, c, and prove that

$$(a+b+c)xyz=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

4. Prove the following identities :

(i) $n^3=n(n-1)(n-2)+3n(n-1)+n$;

(ii) $6n^3=4\{(n+1)^3+(n-1)^3\}-\{(n+2)^3+(n-2)^3\}$.

From (ii), it follows that the cubes of five consecutive numbers are connected by the following relation : "Six times the cube of the middle number increased by the sum of . . . . ." Complete this statement.

5. If $x=3(a+b)$, $y=2a+b$, $z=a+2b$, find the values of the following in terms of a and b :

(i) $(x-y)(x-z)(y+z)\div(xyz)$; (ii) $\sqrt{yz(x-y+z)(x+y-z)}$.

6. What value of x will make the sum of the expressions $\dfrac{x-a}{2(a-b)}$, $\dfrac{x+5b}{3(a+b)}$ equal to 2?

Find the value of each of the expressions when x has this value.

7. A square lawn is surrounded by a path 6 feet wide; the area of the path is 1008 square feet. Find the area of the lawn.

8. The hold of a ship contained 442 gallons of water. This was emptied by two buckets, the greater of which, holding twice as much as the other, was emptied twice in 3 minutes, and the less three times in 2 minutes. If the whole time of emptying was 12 minutes, find the size of each bucket.

13. (To § 105.)

1. If a, b stand for natural numbers, (i) explain why it is not possible to find the value of $a+(-b)$ by a process of counting.

(ii) Explain the method by which a meaning is assigned to $a+(-b)$.

2. Assuming the laws of addition and subtraction for positive numbers, and the definitions of addition and subtraction for negative numbers, prove that $a+(b+c)=a+b+c$ in the case when a and c are positive and b, $(b+c)$ are negative.

3. Simplify the following:

$$\text{(i)}\ a-[3b-\{a-4(c-b)+3c-(2a-\overline{b+c})\}];$$
$$\text{(ii)}\ 12(3a-4b)-3[c-4\{2a-b+c-5(a-b+2c)\}];$$
$$\text{(iii)}\ a-6[a-4\{3a-(2a-\overline{1-a})\}].$$

4. Add together $5(a-2b)+b\{3-5(c-2)\}$,
$5(b-2)(c-3)$ and $(a+8)(c-4)-(a-2)(c+1)$.

5. Solve the following equations, and verify the solutions:

$$\text{(i)}\ \tfrac{11}{7}(5x+3)+\tfrac{2}{9}(13x+8)-\tfrac{3}{5}(7x-11)=0;$$
$$\text{(ii)}\ 4x+5y+z=6,\quad x+7y+2z=10,\quad 5x-3y-6z=16;$$
$$\text{(iii)}\ x+y+z=2a,\quad x-y-z=2b,\quad y-z=a+b,$$

where x, y, z are the unknowns.

6. If $ax-by=2(a^2+b^2)=2(a-b)x-2(a+b)y$, find the values of x and y in terms of a and b, and prove that $x^2+y^2=5(a^2+b^2)$.

7. When x and y have in succession the following values:

(i) $x=10$, $y=2$; (ii) $x=2$, $y=3$; (iii) $x=8$, $y=7$;

the corresponding values of the expression $x^2+y^2+2gx+2fy+c$ are 216, 71, 259. Find the values of g, f, c.

8. If the number of men employed to do a piece of work is increased by 3, the time required for the work will be diminished by 2 days; if the number of men is increased by 12, the time will be diminished by 5 days. Find the number of men employed, and the time required.

14. (To § 122.)

1. (i) The definition of "multiplication by a natural number" assigns no meaning to such an expression as $3 \times (-4)$: explain this.

(ii) How is the operation of multiplying by a negative number defined? Explain why the operation is defined in this way.

2. Explain the following statements:

(i) "To speak of (-5) sovereigns is absurd."

(ii) "If a gain of £5 is denoted by $(+5)$, then a loss of £5 may be denoted by (-5)."

3. Deduce the expanded form of $(x-2y-5z)^2$ from that of

$$(a+b+c)^2.$$

Explain and justify the process. Check your result by putting $x=1$, $y=1$, $z=-1$.

4. Solve the following equations, and verify the solutions:

(i) $\frac{1}{3}(5x-2)+\frac{1}{5}(3x-5)-\frac{1}{2}(3x+1)-\frac{1}{4}(5x+1)=0$;

(ii) $\frac{1}{2}(x+1)(x-2)-\frac{1}{3}x(x+3)-\frac{1}{6}(x-3)(x-5)=0$;

(iii) $\dfrac{2x-4y}{13}-\dfrac{8x+3y}{3}=\dfrac{x+2y}{7}=-1.$

5. Prove the identity

$$(b+c)(c+a)(a+b)=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc,$$

and deduce the expansion of $(x-2y)(x-3z)(2y+3z)$.

6. Obtain the expanded form of $(x+a)(x+b)(x+c)$, and deduce the expansion of $(1-a)(1+2a)(1-3a)$.

7. If $a=-2$, $b=-1$, $c=-5$, find the values of

(i) $\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(b-c)(c-a)(a-b)}$; (ii) $\sqrt{(a^3+2)^2+(b^3-9)^2+c^2(a+b)^2}$.

8. A and B are travelling in the same direction along the same road, A at u miles an hour and B at v miles an hour, where $u>v$. At noon B is a miles in front of A. When does A overtake B?

State questions of the same kind whose solutions are obtained by putting (i) $a=14$, $u=4$, $v=-3$; (ii) $a=2$, $u=3$, $v=4$.

Give the answers to these questions and the necessary explanations.

15. (To § 125.)

1. If $a=2$, $b=-1$, $c=0$, find the values of

(i) $(a+3b-2c)^2-(a^2+3b^2-2c^2)$;

(ii) $(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)$.

2. Find the H.C.F. of

(i) $15x^2-60y^2,\ 3x^2-3xy-18y^2,\ 6x^2+2xy-20y^2$;

(ii) $16x-15(1+x)(1-x),\ (3x+1)^2-4(x-2)^2,\ 5x^3-3x^2-10x+6$.

3. Find the L.C.M. of

(i) $12x^2+8x-15,\ 6x^2+7x-3,\ 18x^2-21x+5$;

(ii) $\{2x^2-x(a-2b)-ab\}^2,\ 16x^4-8x^2a^2+a^4,\ x^4-2x^2b^2+b^4$.

4. Find the square root of the following expressions and verify each result by putting $x=1$, $y=1$:

(i) $49x^2y^2-112xy+64$;

(ii) $(2x^2-8y^2)(8x^2+21xy+10y^2)(16x^2-22xy-20y^2)$;

(iii) $25x^2-30xy+9y^2-20x+12y+4$.

5. Solve the following equations and verify the solutions:

(i) $(x+4)(y-3)=xy+11,\ (x-3)(y-4)=xy+1$;

(ii) $4x-3y-z=-1,\ 2x+2y=z+18,\ x+2y=11-z$.

6. Find x and y in terms of a and b from the equations

$$ax+b^2y=a^2,\ by-x=b.$$

If a and b are connected by a certain relation, these equations become identical. What is this relation?

7. A person bought 80 yards of cloth for £20. 11*s.* 8*d.*, some at 6*s.* 3*d.* a yard and the rest at 5*s.* 5*d.* a yard. How many yards of each kind did he buy?

8. A man buys two horses for £90. By selling one for four-fifths of its cost price, and the other for five-fourths of its cost price, he makes a profit of £9 on the whole transaction. Find the cost price of each horse.

16. (To § 134.)

1. Expand the following products and verify the results when $x=1$: (i) $(4x^2-7x+1)(x+1)^3$; (ii) $(3x-2)^2(2x+3)^2$; (iii) $(x+4)(x-1)^4$.

2. (i) By substituting 1 for x and 1 for a, show that the sum of the coefficients in the expansion of $(x+a)^5$ is 2^5.

(ii) Show that the sum of the coefficients in the expansion of $(x+a)^8$ is twice the sum of the coefficients in the expansion of $(x+a)^7$.

3. If I arrange a certain collection of articles in 13 heaps I shall have 7 over, and if in 14 heaps then I shall have 3 over. In each case there is the same number of articles in a heap; how many articles are there?

4. Obtain two identities by dividing $10x^5+3x^4-2x^3-7x^2-4x+4$ by $5x^2-x+2$, arranging the terms of each expression (i) in descending powers of x, (ii) in ascending powers, and obtaining in each case a quotient of 4 terms. Also check each result by putting $x=1$.

5. Find the coefficient of x^4 in the expansion of each of the following: (i) $(2+3x+4x^2+5x^3)(3-4x-5x^2+6x^3)$; (ii) $(1+x+x^2+x^3+x^4)^2$.

6. When a certain number is divided by 11 the remainder is 4, and when the number is divided by 13 the remainder is 1, also the quotient in the first case exceeds by 3 the quotient in the second case. Find the number.

7. Prove the identity

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3=24abc.$$

By putting $a=2$, $b=9$, $c=4$ in this identity, prove that the sum of the cubes of the numbers 3, (-7), (-11), 15 is a perfect cube.

8. In an action between two battleships, A and B, A fired three times as many shells as B. The total number of misses was 7 times the total number of hits. The number of B's misses was 357, but B's hits exceed A's hits by 66. What was the number of shells fired and the number of hits made by each?

17. (To § 146.)

1. (i) Define the meaning of $\frac{a}{b}$ when a is not exactly divisible by b.

(ii) Explain how ordinal rank is assigned to fractions (*i.e.* how a place is assigned to a given fraction in the scale with integers and other fractions). (iii) Explain why every identity, which is true when the letters stand for natural numbers, is also true for all rational values of the letters.

2. Simplify the following:

$$\text{(i)}\ \frac{x^2-x-12}{x^2+x-20};\quad \text{(ii)}\ \frac{(x-2)^2}{(4-x^2)^2};\quad \text{(iii)}\ \frac{x^3+2x^2-x-2}{x^3+3x^2-4x-12};$$

$$\text{(iv)}\ \frac{x^2+xy+y^2}{x^2+2xy+y^2}\times\frac{x^2-xy+y^2}{x^2-2xy+y^2}\div\left(\frac{x^6-y^6}{(x^2-y^2)^3}\right).$$

3. When a certain polynomial N is divided by $(x+3)$ the quotient is Q and the remainder (-10); when Q is divided by $(x+4)$ the remainder is $(+1)$:

(i) Find the remainder when N is divided by $(x^2+7x+12)$.

(ii) Prove that when $x=7$, N is exactly divisible by 110.

4. Simplify the following :

$$\text{(i)}\ \frac{3}{x^2-3xy}+\frac{1}{2xy+6y^2}+\frac{9y^2+x^2}{xy(9y^2-x^2)}\,;$$

$$\text{(ii)}\ \frac{4x-5}{x^2-7x+10}+\frac{3-2x}{x^2-5x+6}-\frac{2x}{x^2-8x+15}.$$

5. If 8 men do a piece of work in 10 days working x hours a day, and 12 men do the same amount in 8 days working y hours a day, prove that $5x=6y$.

6. Solve the following equations and verify the solutions :

(i) $\frac{1}{4}-\frac{x}{10}-\frac{1}{2}\left(3-\frac{x}{2}\right)=\frac{1}{5}\left(11+\frac{x}{4}\right)$; (ii) $\frac{x-4\frac{2}{3}}{3}-\frac{x-3\frac{2}{3}}{12}=\frac{1}{16}\left\{x-\frac{3x-5}{4}\right\}$;

(iii) $2x+\frac{y-2}{5}=20,\ 3y-\frac{x-17}{2}=40$;

(iv) $4x-2y+z=5,\ 3x+4y-7z=11,\ 5x+6z=-7$.

7. Find x in terms of a and b from the equation

$$\frac{x-a}{1+b}-\frac{x-b}{1+a}=b-a.$$

If a and b are connected by a certain relation, the equation becomes an identity. What is this relation ?

8. In a regiment the number of officers was $\frac{2}{27}$ of the whole number of officers and privates ; but after an engagement in which 6 officers and 42 privates were killed, the officers formed $\frac{3}{46}$ of the whole number. Find the original number of officers.

18. (To § 150.)

1. (i) Why is it necessary to *define* the operations of addition, subtraction, multiplication and division for fractions ?

(ii) If $\frac{a}{b}$ is a fraction, prove that $\frac{ax}{bx}=\frac{a}{b}$.

2. If $3x=4,\ 2y=5,\ 6z=-7$, find the values of

$$\text{(i)}\ \frac{x+z}{y-3}+\frac{2x+3y}{2y+5z}\,;\quad \text{(ii)}\ \sqrt{(6x+5y+\tfrac{9}{2})}+\sqrt{(5y+4z+\tfrac{7}{6})}.$$

3. Simplify the following :

$$\text{(i)}\ \frac{x^2-13x+40}{x^2-5x+4}\times\frac{x^2-4x}{x^2-2x-48}\div\frac{x^2-5x}{x^2+5x-6}\,;$$

$$\text{(ii)}\ \frac{x-3}{2x^2-7x+6}-\frac{4x-1}{6x^2-13x+6}+\frac{4}{3x^2-8x+4}\,;\quad \text{(iii)}\ \frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

$$\text{(iv)}\ \left(\frac{x+a}{x-a}+\frac{x+b}{x-b}\right)\div\left(\frac{x+a}{x-a}-\frac{x+b}{x-b}\right).$$

4. If $(m-n)x=1$, $(m^2-n^2)y=1$, $(m+n)z=1$, find in terms of m and n the values of

$$\text{(i)}\ \sqrt{\frac{2m(m-n)}{x+2ny+z}};\quad \text{(ii)}\ \sqrt{(2x^2+8mny^2+2z^2)}.$$

5. Prove the following identities:

$$\text{(i)}\ \frac{x^5+1}{2x^3+3x^2-1}=\frac{1}{2}x^2-\frac{3}{4}x+\frac{9}{8}-\frac{23}{16x}+\frac{57x^2+34x-23}{16x(2x^3+3x^2-1)};$$

$$\text{(ii)}\ \frac{1+x^5}{-1+3x^2+2x^3}=-1-3x^2-2x^3-9x^4-\frac{13x^5+31x^6+18x^7}{1-3x^2-2x^3}.$$

6. Employ the identities of Ex. 5 to find the approximate values of $\dfrac{x^5+1}{2x^3+3x^2-1}$ when x has the values 46, 0·1, 0·001, giving the first result correct to one place of decimals and the others correct to three places.

7. Employ the process of long division to show that $(a+b+c)$ is a factor of $a^3(b-c)+b^3(c-a)+c^3(a-b)$. State the identity which results from the division and show that

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$$

8. One-fourth of the subscribers to a certain fund gave a sovereign apiece, one-fourth of the remainder gave half-a-sovereign apiece and the rest each gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half-a-guinea and half-a-crown respectively, the total increase in the subscriptions would be £2. 10*s*. How many subscribers were there and what was the total amount subscribed?

19. (To § 158.)

1. If the fractions $\dfrac{x}{a}, \dfrac{y}{b}, \dfrac{z}{c}$ are unequal, show that $\dfrac{x+y+z}{a+b+c}$ lies between the greatest and least of the given fractions.

Show also that $\dfrac{x+y-z}{a+b-c}$ does not necessarily lie between the greatest and least.

2. By eliminating two letters in succession, show that

(i) if $x+2y+3z=0$ and $3x+4y+5z=0$, then $2x=-y=2z$;

(ii) if $\left.\begin{aligned}ax+by+cz&=0\\ \text{and}\ x+y+z&=0,\end{aligned}\right\}$ then $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}$;

(iii) if $\left.\begin{aligned}ax+by+cz&=0\\ \text{and}\ lx+my+nz&=0,\end{aligned}\right\}$ then $\dfrac{x}{bn-cm}=\dfrac{y}{cl-an}=\dfrac{z}{am-bl}$.

3. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, prove that $\frac{y+z-x}{a}=\frac{z+x-y}{b}=\frac{x+y-z}{c}$.

4. Regarding a fraction as a measure, (i) How is the fraction $\frac{a}{b}$ defined? (ii) Explain why $\frac{3}{4}$ of $\frac{5}{7}$ of an inch is equal to $\frac{3\times 5}{4\times 7}$ of an inch. (iii) How is the product of the fractions $\frac{a}{b}$ and $\frac{x}{y}$ defined?

5. A man takes £a to France, spends £b on the journey, and then changes the rest into French money at 25 francs the £.

(i) How long will it last him, at c francs a week?

(ii) How much longer would the money have lasted if he had only spent $c-1$ francs a week?

6. Simplify the following:

(i) $\left\{\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)-\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\right\}\div\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\right)$.

(ii) $\left(\frac{1}{2a+b}+\frac{1}{2a-b}-\frac{3a}{4a^2-b^2}\right)\times\frac{4a^2+4ab+b^2}{2a-b}$.

7. (i) Substitute $\frac{a}{a-1}$ for x and $\frac{2a}{2a+1}$ for y in $\frac{xy}{x-y}$, and reduce the result to its simplest form.

(ii) Prove that $\dfrac{1}{x-\dfrac{x^2}{2x+1-\dfrac{(x+1)^2}{2x+3}}}=\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}$.

8. Three men, A, B, and C, working separately, can do a piece of work in a, b, and c days respectively.

(i) In how many days can they do it when they work together?

(ii) If A works by himself for m days, and then, with the help of B and C, finishes the work in x days more, find x in terms of a, b, c, m.

20. (To § 163.)

1. (i) Find x and y in terms of c and k from the equations

$$\frac{x}{3}+\frac{y}{2}=k-\frac{x}{c}=k+\frac{y}{c}.$$

(ii) Under what circumstances do the equations become (α) inconsistent, (β) not independent?

(iii) Find the values of x and y in the following cases:

(α) $k=1$, $c=6{\cdot}001$; (β) $k=0{\cdot}00001$, $c=6{\cdot}001$.

2. Search for a solution of the equation

$$3\left(\frac{x}{3x+2}+\frac{1}{2x-3}\right)-1=\frac{7x+9}{(2x-3)(3x+2)}.$$

Substitute in the given equation any value which you may obtain for x, and if the value is not a solution explain why this is the case.

3. Simplify the following:

(i) $\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2-4x+3}-\dfrac{1}{x^2-3x+2}.$

(ii) $\dfrac{x^3+\dfrac{1}{x^3}}{x^3-\dfrac{1}{x^3}}\left\{x^2+\dfrac{1}{x^2+1}\right\}\div\left\{x^2-1+\dfrac{1}{x^2}\right\}.$

(iii) $\left(\dfrac{y}{z}+\dfrac{z}{y}\right)^2+\left(\dfrac{z}{x}+\dfrac{x}{z}\right)^2+\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^2-\left(\dfrac{y}{z}+\dfrac{z}{y}\right)\left(\dfrac{z}{x}+\dfrac{x}{z}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}\right).$

4. (i) A runs x miles in y minutes; B walks y miles in x hours. How much start can A give B in a mile race, in which A runs and B walks?

(ii) If a men can reap b acres in c days, in how many days will x men, with the assistance of p extra men during the last q days, reap y acres?

5. Solve for x the following equations:

(i) $\dfrac{2x-3}{x+2}=\dfrac{2x-5}{x-1}.$ (ii) $\dfrac{1}{6}\left(\dfrac{2}{3}x+4\right)+\dfrac{2x-15}{18}=\dfrac{x}{6}\left(\dfrac{6}{x}-1\right).$

(iii) $\dfrac{2a}{x-\dfrac{a}{6}}=\dfrac{3b}{x-\dfrac{b}{3}}.$ (iv) $\dfrac{3x+2}{x+1}+\dfrac{3x+7}{2x-10}-\dfrac{1}{2}=\dfrac{2x+5}{x-5}+\dfrac{7x+1}{3x+3}-\dfrac{1}{3}.$

6. Prove the following identity:

$$a^2-3(a+b)^2+3(a+2b)^2-(a+3b)^2=0.$$

Deduce a theorem relating to (i) four consecutive odd numbers, (ii) four consecutive even numbers.

7. Find in its simplest form the value of

$$\frac{x+y-1}{x-y+1}\text{ when }x=\frac{a+1}{ab+1},\ y=\frac{ab+a}{ab+1}.$$

8. A steamer takes 4 hours less to travel down stream from A to B than from B to A. The steamer travels in still water at 15 miles an hour, and the stream runs at $3\frac{3}{4}$ miles an hour. Find the time of each journey and the distance from A to B.

21. (To § 177.)

1. (i) If the product xyz is zero, prove that one of the factors x, y or z is zero.

(ii) When are two equations said to be equivalent? Explain why the equations $x=3$ and $x^2=9$ are not equivalent.

2. Solve the following equations:

(i) $8x^3+12x^2-2x-3=0$; (ii) $x^2(x+1)(x-2)=9(x+1)(x-2)$;
(iii) $15x^2-68x+60=0$; (iv) $(2x-1)^2=\frac{1}{12}(2x-1)+\frac{1}{2}$.

3. Find equations whose roots are the following, in each case giving the result in its expanded form and free from fractions:

$$\text{(i) } 2,\ 3,\ -5;\quad \text{(ii) } \frac{1}{2},\ \frac{1}{3},\ -\frac{1}{5};\quad \text{(iii) } \frac{c}{a+b},\ \frac{c}{a-b}.$$

4. Given that $(x+2)(2x-1)$ is a factor of

$$24x^4+22x^3-65x^2-16x+20,$$

express the given polynomial as the product of four factors, and find all the roots of the equation $24x^4+22x^3-65x^2-16x+20=0$.

5. Find in terms of y the value of $2x^3-3x^2-3x-5$ when $(y+3)$ is substituted for x. Hence show that for all values of x which are not less than 3, $2x^3>3x^2+3x+5$.

6. A boatman can row at the rate of a miles an hour in still water. He rows a distance of x miles down a river which flows at the rate of b miles an hour, and back again. (i) How long does he take? (ii) Show that the time taken is longer than that which he would require in order to row $2x$ miles in still water.

7. Reduce to their simplest forms

$$\frac{x+y}{1-xy} \text{ and } \frac{x-y}{1+xy}, \text{ where } x=\frac{a+b}{1-ab},\ y=\frac{a-b}{1+ab}.$$

8. Two men A and B cycle north from P, each travelling at a uniform speed. A starts at 1 p.m. and B at 2 p.m. B passes A at 4 p.m., and at 5 p.m. A increases his speed by 5 miles an hour and passes B at 7 p.m. Find the speeds with which A and B commence the journey.

22. (To § 182.)

1. Solve the following equations:

(i) $x^2-60x=5184$; (ii) $x(1{\cdot}8-x)=0{\cdot}7776$;
(iii) $4(x-1)(x+3)-8(x-3)(x+5)+9(x-5)(x+1)=0$.

2. Find two values of a such that the equation
$$(x-a)(2x+a)=(x-2a)(x-a)-51$$
may be satisfied when $x=2$: find also the other values of x which satisfy the given equation when a has these values in succession.

3. Find two values of x for which $2a^3-2a^2+3ax-9x^2$ is exactly divisible by $a-2$: find also the quotients when x has these values in succession.

4. An officer can form his men in a solid square, and also in a hollow square three deep, the number of men in the front row of the hollow square being three more than that in the front row of the solid square. Find the number of men.

5. Simplify the following:

(i) $\dfrac{x-1}{x^2-5x+6}-\dfrac{x-14}{x^2+2x-8}+\dfrac{x-17}{x^2+x-12}$.

(ii) $\dfrac{x^2-ax-2a^2}{x^2+ax-5cx-5ac}\times\dfrac{x^2+2bx-5cx-10bc}{x^2-2ax+2cx-4ac}\times\dfrac{x+2c}{x+2b}$.

(iii) $\left(\dfrac{3x-y}{2x^2-xy-y^2}-\dfrac{3x+y}{2x^2+3xy+y^2}\right)\times\left(\dfrac{x^3-y^3}{xy}+x-y\right)$.

6. Solve for x the following equations:

(i) $\dfrac{15}{x+2}-\dfrac{8}{x-1}=\dfrac{3}{x-4}$; (ii) $\dfrac{2x}{x-5}+\dfrac{3}{5x-2}=\dfrac{6x}{(x-5)(5x-2)}$;

(iii) $2\left(\dfrac{4x+1}{4x-1}\right)^2+5\left(\dfrac{4x+1}{4x-1}\right)=3$; (iv) $pq(ax^2+b)+(p^2a+q^2b)x=0$.

7. ABC is a right-angled triangle of which the hypotenuse BC exceeds the side AC by 8 inches, and AC exceeds $2AB$ by 5 inches. Find the lengths of the sides.

8. A and B ride along the same road, starting together, at 8 and 10 miles an hour respectively. A quarter of an hour later, C pursues them and catches B 26 minutes after catching A. At what rate was C riding?

23. (Graphs.)

1. Plot the points $A(4, 35)$, $B(60, 2)$, $C(24, 50)$, taking one-tenth of an inch as the unit of length along OX and OY. Find by calculation the lengths of BC, CA, AB, and prove that ACB is a right-angle.

2. Plot the points $A(5, 4)$, $B(14, 7)$, $C(8, 15)$. Find by calculation (i) the area of the triangle ABC; (ii) the length of BC; (iii) the length of the perpendicular from A to BC.

3. Plot the points $A(12, 16)$, $B(3, 11)$, $C(2, 3)$.

(i) Find the equations to AB, BC.

(ii) Find the equations to the straight lines CD, AD drawn through C and A respectively parallel to BA and BC.

(iii) By solving the equations obtained in (ii), find the coordinates of D, and verify the result by completing the parallelogram $ABCD$.

4. The average yearly price in pounds per ton of copper for the years 1885 to 1895 is given in the following table:

Year - - - -	'85	'86	'87	'88	'89	'90	'91	'92	'93	'94	'95
Price in pounds per ton	49	45	48	80	55	62	57	50	48	44	47

Exhibit the results graphically, and explain the reason for the following statement:—"If the price for the year 1888 were not given, the rest of the information would not enable us to estimate the probable price for that year."

5. The average yearly price in pounds per ton of pig iron and coal for various years is given below:

Year.	Price of Pig Iron per ton.	Price of Coal per ton.
1882	£2. 2*s*. 7*d*.	5*s*. 8*d*.
1887	£1. 12*s*. 11*d*.	4*s*. 11*d*.
1889	£1. 18*s*. 2*d*.	6*s*. 3*d*.
1890	£2. 7*s*. 8*d*.	8*s*. 1*d*.
1894	£1. 15*s*. 3*d*.	5*s*. 11*d*.
1897	£2. 0*s*. 1*d*.	5*s*. 11*d*.
1900	£3. 8*s*. 1*d*.	10*s*. 6*d*.
1902	£2. 6*s*. 10*d*.	8*s*. 1*d*.

Draw graphs referred to the same axes to show the variation in price of pig iron and coal.

6. If the rate of exchange between England and the United States is 67 dollars to 279 shillings, draw a graph to show the equivalent in dollars of any sum of English money up to £5. Read off (i) the value in English money (correct to 6*d*.) of 9 dollars 40 cents, (ii) the value in American money (correct to 10 cents) of £3. 12*s*.

7. The time T seconds of the vibration of a pendulum of length L inches, for various values of L, is given in the following table:

L	6	12	18	24	30	36	48
T	0·39	0·55	0·68	0·78	0·88	0·96	1·11

Plot the points which represent corresponding values of L and T, and draw a smooth curve through the points. Estimate to the nearest inch the length of a pendulum which beats seconds.

24. (Graphs.)

1. Plot the points $P(7, 4)$, $Q(-5, -2)$, $R(-3, -8)$, $S(9, -2)$.
   (i) What kind of figure is obtained by joining PQ, QR, RS, SP?
   (ii) Find the coordinates of the middle points of PR and QS.
   (iii) State a geometrical theorem of which the result in (ii) is a verification.

2. If O is the intersection of the diagonals of the parallelogram $PQRS$ of Ex. 1, calculate the areas of the triangles POQ, QOR, ROS, SOP.

3. Plot the points $P(2, 5)$, $Q(-6, -1)$, $R(-2, 7)$.
   (i) Find the equations to the perpendicular bisectors of QR, RP, PQ.
   (ii) By means of the equations obtained in (i), prove that the perpendicular bisectors meet in a point O, and find its coordinates.
   (iii) Calculate the lengths of OP, OQ, OR.

4. Plot the points in which the lines whose equations are

$$3x+4y=60, \quad 3x+4y=24,$$

cut the axes, and draw the lines.
   (i) Prove that the points $P(-12, 15)$, $Q(4, 3)$, $R(16, -6)$ all lie on the second line, and verify by plotting the points.
   (ii) If the first line cuts OX, OY respectively in A, B, calculate the areas of the triangles PAB, QAB, RAB.
   (iii) State a geometrical theorem of which the result in (ii) is a verification.

5. The freezing and boiling points of water are marked 0° and 100° Centigrade, and 32° and 212° Fahrenheit. Find the temperature for which the readings on the Centigrade and Fahrenheit scales are the same, and illustrate graphically.

6. The total cost to a refreshment contractor of providing a three-shilling dinner is found to be £$(a+bN)$, where a and b are constants and N is the number of guests.

(i) Prove that if £P is the total profit for N guests, then P and N are connected by a linear relation.

(ii) If the profits and numbers of guests on two days are as below:

Number of guests	2500	3500
Profits - -	£12. 10s.	£37. 10s.

plot the graph which represents the relation between N and P.

(iii) How many guests are necessary just to pay expenses? What profit will be made on 4500 guests? What is the value of a?

7. A burglar, having broken into a house, is interrupted and makes off at 4 a.m. towards a town 15 miles away, alternately running for quarter of an hour and walking for quarter of an hour, at the rates of 10 and 4 miles respectively. The owner of the house, having ascertained the direction in which the thief had gone, pursued him on a bicycle, at the rate of 15 miles an hour, starting at 4.50 a.m. Was the burglar caught before reaching the town; if so, at what time and how far from the house?

25. (Graphs.)

1. Plot the points $P(2, 7)$, $Q(-5, 13)$, $R(-3, -2)$.

(i) Find the coordinates of D, E, F, the middle points QR, RP, PQ respectively.

(ii) Find the equations to PD, QE, RF.

(iii) By considering the equations to PD, QE, RF, prove that these lines meet in a point (G), and find the coordinates of this point.

(iv) State a geometrical theorem of which the result in (iii) is an instance.

2. Plot the points $P(8, 1)$, $Q(-2, 7)$, $R(-8, -8)$.
   (i) Find the coordinates of S, T, the middle points of PQ, PR respectively.
   (ii) Find the equations to ST, QR.
   (iii) By means of the equations to ST and QR, prove that these lines are parallel.
   (iv) State a geometrical theorem of which the result in (iii) is an instance.

3. Plot the points $A(10, 10)$, $B(6, -9)$, $C(-15, 5)$.
   (i) Find the equations to BC, CA, AB.
   (ii) Find the equations to the straight lines through A, B, C respectively perpendicular to BC, CA, AB.
   (iii) By means of the equations to the perpendiculars in (ii), prove that these three lines meet in a point, and find its coordinates.
   (iv) State a geometrical theorem of which the result in (iii) is an instance.

4. Draw graphs of the functions

$$2{\cdot}3\left(\frac{x}{3{\cdot}5} - 1{\cdot}7\right) \text{ and } \frac{1{\cdot}5 - 3x}{2{\cdot}4}.$$

Read off, to one place of decimals, the value of x for which the functions have equal values.

5. Draw, as a "Ready Reckoner," on as large a scale as possible, a graph showing the relation between pence per pint and francs per litre: given that 25 francs = £1 and 1 litre = $1\frac{3}{4}$ pints. Read off from the graph (i) the French equivalent of (*a*) $4\frac{1}{2}d.$ a pint, (*b*) 1*s.* 10*d.* a gallon; (ii) the English equivalent of 3 frs. per litre.

6. Approximate values of a quantity y, corresponding to certain values of another quantity x, are found by experiment to be as shown in the following table:

x	1	1·8	2·8	3·9	5·1	6·0
y	0·223	0·327	0·525	0·730	0·910	1·095

Plot points to represent the pairs of values of x and y. Assuming that x and y are connected by an equation of the form $y = mx + p$, where m and p are constants, find the most probable values of m and p.

7. (i) Find the coordinates of the point (P) of intersection of the lines whose equations are

$$x+2y=3\,; \quad x+(2+k)y=4.$$

(ii) If k diminishes and approaches zero, what happens to the coordinates of P?

(iii) Illustrate by drawing the lines when k has successively the values 0·2, 0·1.

(iv) What kind of lines are represented by inconsistent linear equations?

INDEX

ANSWERS.

EXERCISE I.

p. 6. **1.** 35. **2.** 12. **3.** 12. **4.** 17. **5.** 23. **6.** 35.

p. 7. **7.** 3. **8.** 4. **9.** 5. **10.** 7. **11.** 7. **12.** 27.
13. $20+x$. **14.** $x+20$. **15.** $x+y$.
16. 8. **17.** 4. **18.** 4.
19. $10+6+x$ or $16+x$ miles. **20.** $x+y+4$ miles.
21. $x+y+z$ miles. **22.** $3x+2x+x$ or $6x$ miles. **23.** 5.
24. 5. **25.** 5 miles. **26.** $20=17+3$.
27. $s=a+b$. **28.** $3a$ feet, $36a$ inches. **29.** $1760x$ yards.
30. $8x$ half-crowns, $10x$ florins, $40x$ sixpences.
31. $12x+6y$ pence. **32.** $30x+24$ pence.
33. $3a$ eggs. **34.** 36 eggs. **35.** 16.

p. 8. **36.** $a+b$ apples. **37.** $a+a+b$ or $2a+b$ apples.
38. $x=2a+b$. **39.** $a+6$ years. **40.** $a+y$ years.
41. x years hence. **42.** $x+7$ years.
43. $x+7+3$ or $x+10$ years. **44.** $x+7+y$ or $x+y+7$ years.
45. $2x+y$ years. **46.** $x+2$, $x+3$, $x+4$.
47. $2x+1$. **48.** 22.
49. An even number. **50.** An odd number.
51. $2n$, $2n+2$, $2n+4$. **52.** $2n+1$, $2n+3$, $2n+5$.
53. 20, 4.
54. If to five times a certain number the number 2 is added, the sum is 22: what is the number?
55. $50+x$ pounds, $50+10x$ pounds. **56.** 5.

EXERCISE II.

p. 12. **1.** $x+x$. **2.** $x\times x$. **3.** $x+x+x$. **4.** $x\times x\times x$.
5. x^3+x^3 or $x\times x\times x+x\times x\times x$.
6. $x^2+x^2+x^2$ or $x\times x+x\times x+x\times x$.

p. 12. **7.** 2, 4. **8.** 1, 4. **9.** 3, 6. **10.** 1, 8.
11. 2, 16. **12.** 3, 12. **13.** $ab+ab$.
14. Multiply a by b, then multiply the product by c.
15. 6. **16.** 6. **17.** 6. **18.** 18.
19. 18. **20.** $20x$ pence. **21.** ax pence.
22. $112x$ pence. **23.** $112ax$ pence. **24.** $ax+by$ pence.
25. $3v$ miles, vt miles. **26.** $s=vt$.
27. If a and b stand for any numbers, then $a \times b = b \times a$.
28. 6. **29.** 3. **30.** 3.
31. 13. **32.** 35. **33.** 15. **34.** 45.
35. 225. **36.** 10. **37.** 28. **38.** 1.
39. 3. **40.** 11. **41.** 125. **42.** 5.

p. 13. **43.** (i) $4x$ miles, (ii) 7. **44.** (i) $4x=3y$, (ii) 6, (iii) 12.
45. (i) 880, (ii) 44, (iii) 6.
46. (i) 16, 144, 1600 feet; (ii) 4, 2; (iii) 5 seconds.
47. (i) 323, (ii) 16, (iii) 25.
48. (i) x^2 sq. ft., (ii) x sq. ft., (iii) $21x$ sq. ft., (iv) ph sq. ft. (v) $70h$ sq. ft., (vi) $2lh+2bh$ sq. ft.

p. 14. **49.** (i) 7, (ii) 4, (iii) 1. **50.** (i) 5, (ii) 3, (iii) 1.
53. 12, 1; 6, 2; 4, 3; 3, 4; 2, 6; 1, 12.
54. $x=3$, $y=4$ and $x=4$, $y=3$.
55. (i) 8, (ii) 4. **56.** $xy=27$; $x=3$, $y=9$ and $x=9$, $y=3$.
57. 5 and 6. **58.** 3 and 8. **59.** 2 and 9.

EXERCISE III.

p. 18. **1.** 12. **2.** 27. **3.** 15. **4.** 44. **5.** 21.
6. 10. **7.** 29. **8.** 40. **9.** 40. **10.** 25.
11. 94. **12.** 34. **13.** 64. **14.** 99. **15.** 55.
18. $x=2a+3b+4c$. **19.** (i) 31, (ii) 49.
20. $x=4a+3b+26$; 125 miles.
21. $x=2a+4b+2c+d$; 1450 pounds.
22. $x=a+2b+3c+d+5$; 95 runs.

p. 19. **23.** 42 years. **24.** $48x$ shillings; 2 shillings.
25. $6y+75$ shillings; £1. 10*s*. **26.** $10a+15b+5c$ feet; 5.
27. $17t$ feet, $16t$ feet; 160 seconds.
28. (i) $a+11d$, $a+19d$; (ii) 23, 39; (iii) 24, 40; (iv) 35, 59.
29. 1999. **30.** (i) 4, (ii) 9. **31.** (i) 61, (ii) 43, (iii) 102.

EXERCISE IV.

p. 23. **1.** 9. **2.** 14. **3.** 10. **4.** 14. **5.** 300. **6.** 200. **7.** 60. **8.** 2300. **9.** 20. **10.** 11. **11.** 20. **12.** 14. **13.** 8. **14.** 36. **15.** 72. **16.** 100. **17.** $x(x+1)$. **18.** $y(x^2+2)$. **19.** $c(b+a+3)$. **20.** $x(x^2+x+1)$. **21.** $c(ab+c^2)$. **22.** $(x+y)(x+y)$ or $(x+y)^2$. **23.** $(x+y)(x+y+1)$. **24.** $(x+1)(x+3)$. **25.** $(x+2y)(x+y)$. **26.** $(x+2y^2)$. **27.** $15a$. **28.** $4a$. **29.** $16a$. **30.** $3a+b$. **31.** $3a+4b$. **32.** $5a+8b+8c$. **33.** $x+4y$. **34.** $6x+6y+9z$. **35.** $2x+12$. **36.** $10x+9$. **37.** $10x$. **38.** $5x+5y$. **39.** $4x+5y+6z$. **40.** $2x+2y+2z$. **41.** $4x+4y+4z$. **42.** $2x+4y+2z$. **43.** $11x+10y+2z$.

p. 24. **44.** $7a+9b+2c$. **45.** $4a+5b+6c$. **56.** (i) $bc+a$, (ii) $(a+b)(c+d)$. **57.** 5 ; 3. **58.** 6. **59.** $x=7$, $y=1$. **60.** (i) $v=175$, $s=475$; (ii) $u=10$, $s=174$; (iii) $t=2$, $s=88$; (iv) 276 ft. **61.** (i) $4z=ax+by$; (ii) 6 ; (iii) $1\frac{1}{2}d.$, $\frac{3}{4}d.$, $\frac{1}{2}d.$, $\frac{1}{4}d.$ each. **62.** (i) $24=xt+yt$; (ii) 4 p.m.

EXERCISE V.

p. 27. **1.** 3. **2.** 3. **3.** 7. **4.** 7. **5.** 1. **6.** 10. **7.** 7. **8.** 1. **9.** 5. **10.** 1. **11.** 4. **12.** 1. **13.** 7. **14.** 10. **15.** 3. **16.** 5. **17.** 5. **18.** 5. **19.** 5. **20.** 8. **21.** 2. **22.** 8. **23.** 18. **24.** 15. **25.** 27. **26.** 3. **27.** 16.

p. 28. **29.** (i) $a\times(b\div c)$; (ii) $a\times(c-b)$; (iii) $a\div b\div c$; (iv) $a\div(cb)$; (v) $a\div(c+b)$. **43.** $x-20$ years. **44.** $40-x$ years. **45.** $y-x$ years. **46.** 5. **47.** $100-ax$ miles ; 14 miles an hour. **48.** $240x+12y-z$ pence. **49.** $x=ut-vt$; 40 yds.

p. 29. **50.** (i) $v=2680$, $s=28,400$; (ii) $t=62$, $s=62,496$; (iii) $u=2277$, $s=81,011$; (iv) 85 seconds, 139,400 feet. **51.** (i) 11, 15, 19 ; (ii) 13, 14, 15 ; (iii) 6 ; (iv) 3. **52.** (i) $3x-4y$ miles ; (ii) $3x-4y=5$; (iii) 11, 15, 19 ; (iv) 16, 19, 22 ; (v) 3. **53.** (i) $xt-yt=3a$; 15.

EXERCISE VI.

p. 32. **1.** $2a$. **2.** x. **3.** x. **4.** $2a$. **5.** $2a$.
6. $a^2-2ab+b^2$. **7.** $4a-b$. **8.** $a^2-2ab-b^2$.
9. $3a-5b$. **10.** $a-5$. **11.** b. **12.** $a-b$.
13. $n-4$. **14.** $n-2$. **15.** 2. **16.** $2n-2$.
17. $n+2$. **18.** $n-12$. **19.** a. **20.** $a-2b$.
21. $2a-b$. **22.** b. **23.** $4x$. **24.** $2x+2y$.
25. Rises 400 ft., rises 500 ft., sinks 800 ft.
26. (*a*) spend 12*d*., earn 6*d*., spend 3*d*.;
(*b*) spend 12*d*., spend 3*d*., earn 6*d*.;
(*c*) spend 3*d*., spend 12*d*., earn 6*d*.: 5*d*. must be borrowed.
27. $100-ax-by$ miles. **28.** $a-(b-c)$ feet, $a-b+c$ feet.

p. 33. **29.** $60-ax-by$ pence.
30. (i) $x-1$, x, $x+1$; (ii) $y-2$, $y-1$, y.
31. (i) $2n-3$, $2n-1$, $2n+1$; (ii) $2n-7$, $2n-5$, $2n-3$.
32. $6n$. **33.** $3x-3$. **34.** $3x-6$.
35. $5a^2-5b^2+3a-3b$. **36.** $3c-2a-b$.
37. $2x^3+2x^2+2x-10$. **38.** $a+b-2c$; $b+c$.
39. $4x+6y-7z$; $2x+7y-z$.
40. $3x^3+x^2y+xy^2+3y^3$; $3x^3-2x^2y+3xy^2+3y^3$.

EXERCISE VII.

p. 39. **1.** A is as tall as C. **2.** $y=z$.
3. Subtract 3 from each side; 7. **4.** Art. 24, Rule 5. **5.** 3.
6. If 5 is added to twice a certain number the sum is 11.
7. 5. **8.** 6, 3.
9. If three is added to a certain number, and the sum multiplied by 2, the product is 12.
10. 16. **11.** 2. **12.** 10. **13.** 3. **14.** 7.
15. 11. **16.** 3. **17.** 4. **18.** 7.
19. (1) Associative Law; (2) Commutative Law; (3) Art. 24, Rule 3; (4) Art. 24, Rule 5.
20. The sum of three consecutive numbers is 93; find the numbers.

p. 40. **21.** (i) 38, (ii) 60, (iii) $(x-1)$, (iv) $99-x$, (v) there are as many numbers before the middle term as after it: hence we equate the results obtained in (iii) and (iv).
22. (1) x is added to each side; (2) 1 is added to each side.

23. 78: show that as many of the given numbers come before 78 as after it.
24. 4 and 5.
25. As many of the given numbers come before $x-1$ as follow x.
26. 101; 100, 101. 27. $5x$.
28. (i) £$(y-2x)$; (ii) £$(y+2x)$; (iii) £$5y$.
29. (i) 24; $x=8$, $y=7$. 30. (i) 88; $x=11$, $y=7$.

p. 41. 31. (i) $x=11$, $y+z=10$; (ii) $y=8$, $z=2$.
32. (i) $x+y=8$, $x-y=2$; (ii) $x=5$, $y=3$.
33. (i) 6, 1; 3, 2; 2, 3; 1, 6. (ii) 3, 2; 2, 3.
34. (i) $10x-(21-x)=56$; $x=7$, $y=2$.
35. (i) $p=2(l+b)$; (ii) 22; (iii) 24; (iv) 100.
36. 27, 66. 37. 65, 38. 38. 9, 38. 39. A £133, B £86.
40. A £79, B £21. 41. $4n+2a$ years; $n=2a$.

p. 42. 42. 48, 49. 43. $6n+9$. 44. 109, 111, 113.
45. (i) 89. (ii) A, £1. 2*s*. 3*d*.; B, 14*s*. 10*d*.; C, 7*s*. 5*d*.
46. (i) $22x$ shillings; 30 shillings.
47. (i) £$2x$, £$(x-6)$; (ii) £$(4x-6)$; (iii) £15.
48. 36, 37. 49. 21, 11. 50. 10. 51. $y=5x$.

EXERCISE VIII.

p. 47. 1. $2x^3$. 2. $3x^2$. 3. $3xyz$. 4. $x^3y^3z^3$.
5. $x^2yz+xy^2z+xyz^2$. 6. x^{12}. 7. $35x^{12}$.
8. $6a^3b^4$. 9. $30x^5y^2$. 10. ab^3c^4. 11. $3x^2$.
12. x^6. 13. x^6. 14. x^6. 15. $8x^6$.
16. $9x^6$. 17. $81x^{12}$. 18. $64x^{12}$. 19. $2x^6$.
20. x^{20}. 21. $6x+9y$. 22. $8a^3+20a^2b$.
23. a^3b+ab^3. 24. $a^3b^2+a^2b^3$. 25. $6x^5+2x^2$.
26. $12x^5y^2+9x^3y^4$. 27. $10x^4+15x^3+20x^2$. 28. $x^2+2xy+y^2$.
29. $x(x+1)$. 30. $2x^2(x+1)$. 31. $3x(x^2+2)$.
32. $ab(a+b)$. 33. $xyz(x+y+z)$. 34. $xy(xy+1)$.
35. $5x^2y^2(2y+x)$. 36. $(x+y)(x+y)$. 37. $(x+y)(x+y+1)$.
38. $(x+1)(x+2)$. 39. $(x+y)(x+2y)$. 40. $(x+2y)(x+2y)$.
41. 200. 42. 2000. 43. 25000. 44. 4000000.

p. 48. 46. $2xy+2yz+2zx$. 47. $5xy+5yz+5zx$.
48. (i) $xa+ya+xb+yb$.
49. (i) $16x^2+40xy+25x^2$; (ii) $4x^4+12x^2y^2+9y^4$.

p. 48. 50. (i) x^2+2x+1; (ii) x^2+6x+9; (iii) $x^2+10x+25$; (iv) $4x^2+4x+1$; (v) $16x^2+8x+1$; (vi) $4x^2+12x+9$; (vii) $9x^2+24x+16$; (viii) $25x^2+60xy+36y^2$; (ix) $49x^2+112xy+64y^2$; (x) $x^4+2x^2y^2+y^4$; (xi) $4x^4+12x^2y^2+9y^4$; (xii) $x^6+2x^3y^3+y^6$.

51. (i) $x^3+6x^2+12x+8$; (ii) $27x^3+27x^2+9x+1$; (iii) $8x^3+60x^2y+150xy^2+125y^3$; (iv) $x^6+3x^4y^2+3x^2y^4+y^6$; (v) $x^3y^3+12x^2y^2z^2+48xyz^4+64z^6$.

52. (i) $x^2+7x+12$; (ii) $x^2+11x+30$; (iii) $x^2+22x+120$; (iv) $10x^2+7x+1$; (v) $8x^2+22x+15$; (vi) $35x^2+82x+48$; (vii) x^3+2x^2+2x+1; (viii) $4x^3+6x^2y+6xy^2+9y^3$.

53. $x^4+4x^3+6x^2+4x+1$.

54. (i) $4x^2+12x+9$; (ii) $8x^3+24x^2+24x+8$.

EXERCISE IX.

p. 51. 1. (i) xy, yz, zx; (ii) x^2, xy. 2. (i) x^2y, x^2z, xyz; (ii) x^3, x^2y.

3. x, y, xy. 4. abc^2. 5. $2a$.
6. ay. 7. ax. 8. $3a^2bc$.
9. $12ac$. 10. $a^3b^7c^3$. 11. $6xyz$.
12. py. 13. $3(a+b)$. 14. $a+2b$.
15. $a+b$. 16. abc. 17. $12abxy$.
18. a^2b^2xyz. 19. $a^2b^3c^2$. 20. $12x^2y^2z^2$.
21. $p^6q^6r^7$. 22. $168a^2b^2x^2yz^2$. 23. $240l^6m^4n^5$.
24. $5a^2(a-b)(a+b)$. 25. $6(a+b)^2$. 26. $c^2(a+b)^2$.
27. H.C.F. $=a^4b^2$; L.C.M. $=a^6b^8$.
28. H.C.F. $=a^3b^2c^2$; L.C.M. $=5184a^4b^6c^8$.
29. H.C.F. $=a^6b^6$; L.C.M. $=a^{12}b^{12}$.
30. H.C.F. $=z^8$; L.C.M. $=x^2y^4z^9$.

EXERCISE X.

p. 53. 1. 700. 2. 390. 3. 27.
4. 300. 5. 320. 6. $7(1+2x)$.
7. $5(2x-y)$. 8. $a(2a-b)$. 9. $a^2(a-1)$.
10. $2a^3(a-1)$. 11. $3ab(1+ab)$. 12. $ab(a+b)$.
13. $5a^2b^2(a-b)$. 14. $7ab(b-2)$. 15. $3x^3(3x^6-1)$.
16. $3x(3x-1)$. 17. $x^2y^3z^2(x^2+z^2)$. 18. $xy^2z(z-x)$.
19. $a(a+b+c)$. 20. $2x(4x^2+x-2)$.
21. $xyz(x^2y+y^2z-xz^2)$. 22. $5xy(x^3-2x^2y+2xy^2-y^3)$.

23. $(a+b)(c+d)$. 24. $(a-b)(c-d)$. 25. $(x-b)(x+a)$.
26. $(x+y-z)(a+b)$. 27. $(x-2)(x^2+1)$. 28. $(x^2+1)(x+1)$.
29. $(a-1)(b+1)$. 30. $(x-1)(a-b)$. 31. $(a-b)(a-b-c)$.
32. $(a-b)(a-b-1)$. 33. $x+y$. 34. $8x-7y$.
35. $3xz$. 36. $yz+zx+xy$. 37. $xz^2+yx^2+zy^2$.

p. 54. 38. $4a+17b-8c$. 39. $9b-11c$.
40. $2a^2-3b^2+c^2$; $a^2+b^2+c^2$. 41. $6ax-6b$.
42. (i) $ax+ay-bx-by$; (ii) $ax-ay-bx+by$.

EXERCISE XI.

p. 56. 1. $x^2+7x+10$. 2. $x^2-7x+10$. 3. $x^2-3x-10$.
4. $x^2+3x-10$. 5. $15x^2+8x+1$. 6. $15x^2-8x+1$.
7. $15x^2+2x-1$. 8. $15x^2-2x-1$. 9. $a^2+2ab+b^2$.
10. $a^2-2ab+b^2$. 11. $20x^2+9xy+y^2$. 12. $20x^2-9xy+y^2$.
13. $20x^2+xy-y^2$. 14. $20x^2-xy-y^2$.
15. $8x^2+26xy+21y^2$. 16. $8x^2-26xy+21y^2$.
17. $8x^2-2xy-21y^2$. 18. $8x^2+2xy-21y^2$.
19. x^4+3x^2-40. 20. $6x^2y^2-3xy-30$.
21. $7x^4-x^2y^2-8y^4$. 22. x^3-1. 23. x^3+1.
24. $8x^3-1$. 25. $27x^3+1$.
26. $a^3+3ab+3ab^2+b^3$. 27. a^3-b^3. 28. a^3+b^3.

p. 57. 29. $6x^3-12x^2-10x+20$. 30. x^3-7x-6.
31. x^4-16. 32. $x^4+3x^3-7x^2-27x-18$.
33. $6x^4-17x^3-106x^2+217x-60$.
34. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
35. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. 36. $a^4-2a^2b^2+b^4$.

EXERCISE XII.

p. 59. 1. (i) $(2x+1)^2=4x^2+4x+1$, $(2x-1)^2=4x^2-4x+1$;
(ii) $(5x+3y)^2=25x^2+30xy+9y^2$,
$(5x-3y)^2=25x^2-30xy+9y^2$;
(iii) $(x^2+y^2)^2=x^4+2x^2y^2+y^4$, $(x^2-y^2)=x^4-2x^2y^2+y^4$;
(iv) $(3x^3+y^3)^2=9x^6+6x^3y^3+y^6$, $(3x^3-y^3)^2=9x^6-6x^3y^3+y^6$;
(v) $[x+(y-z)]^2=x^2+2x(y-z)+(y-z)^2$,
$[x-(y-z)]^2=x^2-2x(y-z)+(y-z)^2$;
(vi) $[2x+(3y-5z)]^2=4x^2+4x(3y-5z)+(3y-5z)^2$,
$[2x-(3y-5z)]^2=4x^2-4x(3y-5z)+(3y-5z)^2$.

p. 59. 2. (i) x^2-2x+1; (ii) $x^2-8x+16$; (iii) $x^2+10x+25$;
(iv) $9x^2-6x+1$; (v) $9x^2-12x+4$;
(vi) $25x^2+40xy+16y^2$; (vii) $36x^2-84xy+49y^2$;
(viii) $49x^2+112xy+64y^2$; (ix) $49x^2y^2-28xy+4$;
(x) $9(25x^2+70xy+49)=225x^2+630xy+441$;
(xi) $x^4-2x^2y^2+y^4$; (xii) $36x^6-12x^3+1$.

3. (i) $2x+5$; (ii) $3x-2$; (iii) $4x-y$;
(iv) $2x^2-y$; (v) $6x^3+5y^3$; (vi) $2x+5y-1$.

4. (i) x^2-1; (ii) x^2-25; (iii) $4x^2-9$;
(iv) $16x^2-25$; (v) $25x^2-36y^2$; (vi) x^4-y^4;
(vii) $9x^6-y^6$; (viii) a^2b^2-4; (ix) a^8-b^8.

5. (i) $x^2+y^2+z^2-2xy-2xz+2yz$;
(ii) $x^2+y^2+z^2-2xy+2xz-2yz$.

6. (i) $4x^2+9y^2+z^2+12xy-4xz-6yz$;
(ii) $4x^2+9y^2+16z^2-12xy-16xz+24yz$;
(iii) $x^2+4y^2+25z^2+4xy+10xz+20yz$;
(iv) $x^4+16y^4+z^4-8x^2y^2+2x^2z^2-8y^2z^2$;
(v) $x^4+2x^3+3x^2+2x+1$; (vi) $4x^4-12x^3+5x^2+6x+1$;
(vii) $9x^2-24x^3+46x^2-40x+25$;
(viii) $x^6+2x^5-x^4-4x^3-x^2+2x+1$.

p. 60. 7. (i) $(2x+1)^3=8x^3+12x^2+6x+1$,
$(2x-1)^3=8x^3-12x^2+6x-1$;
(ii) $(5x+3y)^3=125x^3+225x^2y+135xy^2+27y^3$,
$(5x-3y)^3=125x^3-225x^2y+135xy^2-27y^3$;
(iii) $(x^2+y^2)^3=x^6+3x^4y^2+3x^2y^4+y^6$,
$(x^2-y^2)^3=x^6-3x^4y^2+3x^2y^4-y^6$;
(iv) $(3x^3+y^3)^3=27x^9+27x^6y^3+9x^3y^6+y^9$,
$(3x^3-y^3)^3=27x^9-27x^6y^3+9x^3y^6-y^9$.

8. (i) x^3-3x^2+3x-1; (ii) $x^3-12x^2+48x-64$;
(iii) $x^3+15x^2+75x+125$; (iv) $27x^3-27x^2+9x-1$;
(v) $27x^3-54x^2+36x-8$;
(vi) $125x^3+300x^2y+240x^2y+64y^3$;
(vii) $216x^3-756x^2y+882xy^2-343y^3$.

9. (i) $8x^3+1=(2x+1)(4x^2-2x+1)$,
$8x^3-1=(2x-1)(4x^2+2x+1)$;
(ii) $125x^3+27y^3=(5x+3y)(25x^2-15xy+9y^2)$,
$125x^3-27y^3=(5x-3y)(25x^2+15xy+9y^2)$;

(iii) $x^6+y^6=(x^2+y^2)(x^4-x^2y^2+y^4)$,
$x^6-y^6=(x^2-y^2)(x^4+x^2y^2+y^4)$;
(iv) $27x^6+y^6=(3x^2+y^2)(9x^4-3x^2y^2+y^4)$,
$27x^6-y^6=(3x^2-y^2)(9x^4+3x^2y^2+y^4)$.

10. (i) $(7+3)^2=100$; (ii) $(7-6)^2=1$; (iii) $(8+2)^3=1000$; (iv) $(9+1)^3=1000$; (v) $(5-4)^3=1$.

11. x^4+x^2+1. **12.** $a^2x^2-2a^2x+a^2-4$.
13. $x^4-7x^2y^2+y^4$. **14.** $x^4-x^2y^2+4xy^3-4y^4$.
15. a^4+4b^4. **16.** $x^6-4x^4-4x^3-9x^2+16$.
17. $x^4-a^2x^2-b^2x^2+a^2b^2$. **18.** x^4-3x^2+9.
19. $x^5-8x^4y+14x^3y^2+9x^2y^3-6xy^4$. **20.** $x^8+x^4y^4+y^8$.
21. 1. **22.** $4y^2$. **23.** $4x^2$. **24.** $16y^2$.
25. y^3. **26.** $27x^3+81x^2y+81xy^2+27y^3$.

p. 61. **27.** x^6-1. **28.** x^6-y^6. **29.** x^8-y^8. **30.** $3x+7$.
31. $5xy$. **32.** $12xy$. **33.** $4ax-a^2$. **34.** $8xy$.
35. $(a+b)^2x$. **36.** $10x^2y^2$. **39.** $4a^2b^2$.
40. $(a-6)(b-6)$ sq. ft. **41.** $4b(3a+b)$.
42. (i) $6a^2-12a+8$; (ii) $2(bc+ca+ab)-4(a+b+c)+8$.
43. (i) $6a^2$; $2(bc+ca+ab)$;
(ii) $6a^2-24a+24$; $2(bc+ca+ab)-8(a+b+c)+24$.

EXERCISE XIII.

p. 63. **1.** $(b+c)(a+c)$. **2.** $(b+c)(a-c)$. **3.** $(b-c)(a-c)$.
4. $(a+b)(a+2)$. **5.** $(x+y)(x-3)$. **6.** $(3-2z)(x-y)$.
7. $(x-1)(y+2)$. **8.** $(x-3)(y-5)$. **9.** $(x+5a)(2y-3a)$.
10. $(x+a)(x+b)$. **11.** $(x+a)(x-b)$. **12.** $(x-a)(x-b)$.
13. $(y-x)(y+9)$. **14.** $(a-1)(a^2-1)$. **15.** $(x+y)(x^2+y^2)$.
16. $(x-y)(x+y+z)$. **17.** $(ax+y)(x+ay)$.
18. $(al+bm)(bl+am)$. **19.** $(a+b)(a+b+2c)$.
20. $(a+b)(a-b+2c)$. **21.** $(a-b)(a+b-2c)$.
22. $(y-ax^2)(y+bx^2)$. **23.** $(ax+by)(x+y+z)$.
24. $(5+b)(y-ax)$. **25.** $(5-ax)(b-ax)$.
26. $(a+b+c)(ab+bc+ca)$. **27.** $(2xy+3z)(3xz+4y)$.
28. $(a-b)(a+b+c)$. **29.** $x(x+a)(x+b+c)$.
30. $(a+2b)^2(a+2b-c)$. **31.** $(3a-2b)^2(3a-2b-2)$.

EXERCISE XIV.

p. 65. **1.** x^2+3x+2. **2.** $x^2-8x+15$. **3.** $x^2-12x+35$.
4. $x^2+17x+60$. **5.** $x^2-11x+28$. **6.** $25+10x+x^2$.
7. $x^2-5xy+6y^2$. **8.** $x^2+11xy+30y^2$. **9.** $42x^2+13xy+y^2$.

p. 65. **10.** $80x^2-18xy+y^2$. **11.** x^4-10x^2+24. **12.** x^6-15x^3+56.
13. $(x+1)(x+3)$. **14.** $(x-2)(x-3)$. **15.** $(x-1)(x-4)$.
16. $(x+2)^2$. **17.** $(x-2)(x-4)$. **18.** $(x-1)(x-8)$.
19. $(x+3)^2$. **20.** $(x+1)(x+9)$. **21.** $(x+2)(x+8)$.
22. $(x+3)(x+7)$. **23.** $(x+4)(x+6)$. **24.** $(x+5)^2$.
25. $(x-5y)(x-6y)$. **26.** $(x-4y)(x-7y)$.
27. $(x-3y)(x-8y)$. **28.** $(x-2y)(x-9y)$.
29. $(x-y)(x-10y)$. **30.** $(x+y)(20x+y)$.
31. $(2x+y)(10x+y)$. **32.** $(4x+y)(5x+y)$.
33. $(1+x^2)(28+x^2)$. **34.** $(2+x^2)(14+x^2)$.
35. $(4+x^2)(7+x^2)$. **36.** $(xy-1)(xy-24)$.
37. $(xy-2)(xy-12)$. **38.** $(xy-3)(xy-8)$.
39. $(xy-4)(xy-6)$. **40.** $(x^3+1)(x^3+40)$.
41. $(x^3+2)(x^3+20)$. **42.** $(x^3+4)(x^3+10)$.
43. $(x^3+5)(x^3+8)$. **44.** $(x^2+y)(x^2+30y)$.
45. $(x^2+2y)(x^2+15y)$. **46.** $(x^2+3y)(x^2+10y)$.
47. $(x^2+5y)(x^2+6y)$. **48.** $(x^4+10)^2$.
49. $(x+y-z)(x+y-5z)$. **50.** $(z-x-y)(z-5x-5y)$.
51. $(x-y-2z)(x-y-5z)$. **52.** $(z-2x+2y)(z-5x+5y)$.

EXERCISE XV.

p. 67. **1.** x^2+x-2. **2.** x^2-x-2. **3.** $x^2+4x-21$.
4. $x^2-4x-21$. **5.** $x^2-2xy-48y^2$. **6.** $x^2+5xy-50y^2$.
7. x^4-5x^2-24. **8.** x^6+5x^3-36. **9.** $x^2y^2+4xy-96$.
10. $(x+3)(x-1)$. **11.** $(x+4)(x-2)$. **12.** $(x+5)(x-3)$.
13. $(x-4)(x+1)$. **14.** $(x-5)(x+2)$. **15.** $(x-6)(x+3)$.
16. $(x+12)(x-1)$. **17.** $(x-12)(x+1)$. **18.** $(x-6)(x+2)$.
19. $(x+6)(x-2)$. **20.** $(x+4)(x-3)$. **21.** $(x-4)(x+3)$.
22. $(x+24)(x-1)$. **23.** $(x-12)(x+2)$. **24.** $(x-8)(x+3)$.
25. $(x+6)(x-4)$. **26.** $(x+18)(x-1)$.
27. $(x-9)(x+2)$. **28.** $(x-6)(x+3)$.
29. $(x+40y)(x-y)$. **30.** $(x-20y)(x+2y)$.
31. $(x-10y)(x+4y)$. **32.** $(x+8y)(x-5y)$.
33. $(x^3-28)(x^3+1)$. **34.** $(x^3-14)(x^3+2)$.
35. $(x^3+7)(x^3-4)$. **36.** $(x^2-30)(x^2+1)$.
37. $(x^2-15)(x^2+2)$. **38.** $(x^2+10)(x^2-3)$.
39. $(x^2+6)(x^2-5)$. **40.** $(x-15)(x+10)$.
41. $(x+20)(x-12)$. **42.** $(x-35)(x+7)$.
43. $(x-30y^2)(x+10y^2)$. **44.** $(x-24y)(x+15y)$.

45. $(x+28)(x-8)$. 46. $(x+24)(x-22)$.
47. $(x-16)(x+15)$. 48. $(x-24y^4)(x+20y^4)$.
p. 68. 49. $(x+14y)(x-8y)$. 50. $(x-21y^6)(x+15y^6)$.
51. $(x-27y)(x+15y)$. 52. $(5x-3y)(3y-x)$.
53. $4(3x+5y)(x-y)$. 54. $4y(15x-2y)$.
55. $(x+y+5z)(x+y)$. 56. $(x+y+5z)(x+y)$.

EXERCISE XVI.

p. 70. 1. $2x^2+5x+3$. 2. $2x^2+7x+3$. 3. $3x^2+13x+4$.
4. $3x^2+8x+4$. 5. $3x^2+7x+4$. 6. $5x^2-31x+6$.
7. $5x^2-17x+6$. 8. $5x^2-13x+6$. 9. $5x^2-11x+6$.
10. $6x^2+13x+6$. 11. $6x^4+13x^2+5$. 12. $6x^4+17x^2+5$.
13. $6x^4+31x^2+5$. 14. $6x^4+11x^2+5$. 15. $10x^2-29xy+21y^2$.
16. $10x^2-41xy+21y^2$. 17. $10x^2-73xy+21y^2$.
18. $10x^2-37xy+21y^2$. 19. $10x^2-107xy+21y^2$.
20. $10x^2-47xy+21y^2$. 21. $ax^2+(a+b)x+b$.
22. $ax^2+(ab+1)x+b$. 23. $2ax^2-(a+2b)x+b$.
24. $2ax^2-(ab+2)x+b$. 25. $6acx^2+(2ad+3bc)x+bd$.
26. $(x+1)(3x+2)$. 27. $(x-1)(3x-2)$. 28. $(x-2)(3x-1)$.
29. $3(x+1)(x+2)$. 30. $(3x+1)(x+6)$. 31. $(x+3)(3x+2)$.
32. $(x+3)(2x+1)$. 33. $(x+1)(2x+3)$. 34. $(x-2)(2x-3)$.
35. $(x-6)(2x-1)$. 36. $(x+2)(3x+2)$. 37. $(x+1)(3x+4)$.
p. 71. 38. $(x+4)(3x+1)$. 39. $(x-1)(5x-3)$. 40. $(x-3)(5x-1)$.
41. $(x-3)(3x-5)$. 42. $(x-15)(3x-1)$. 43. $(x-2)(3x-5)$.
44. $(x-5)(3x-2)$. 45. $(x-10)(3x-1)$. 46. $(x-1)(3x-10)$.
47. $(x+3)(5x+2)$. 48. $(x+2)(5x+3)$. 49. $(x+1)(5x+6)$.
50. $(x+6)(5x+1)$. 51. $(2x+1)(3x+7)$.
52. $(x+7)(6x+1)$. 53. $(3x-2y)(7x-y)$.
54. $(x-y)(21x-2y)$. 55. $(5x+y)(3x+7y)$.
56. $(x+7y)(15x+y)$. 57. $(2x-1)(5x-7)$.
58. $(x-7)(10x-1)$. 59. $(2x-1)(2x-5)$.
60. $(x^2-1)(4x^2-5)$. 61. $(2xy-3)(5xy-6)$.
62. $(xy-6)(10xy-3)$. 63. $(2x+3)(5x+7)$.
64. $(x+3)(10x+7)$. 65. $5(x+2)(6x+1)$.
66. $3(3x-2)(7x-3)$. 67. $4(3x+5)(\ x+4)$.
68. $(4x+5)(5x+6)$. 69. $(7x-8)(2x-7)$.
70. $(2x-1)(8x-3)$. 71. $(7x+1)(14x+5)$.
72. $(6x-1)(12x-5)$. 73. $(11x-2)(22x-3)$.
74. $(20x+9)(30x+7)$. 75. $(9x-2)(18x-5)$.

EXERCISE XVII.

p. 73.

1. $2x^2+x-3$. 2. $2x^2-x-3$. 3. $2x^2+5x-3$.
4. $2x^2-5x-3$. 5. $3x^2-11x-4$. 6. $3x^2+4x-4$.
7. $3x^2+x-4$. 8. $6x^2+5x-6$. 9. $6x^2-7x-5$.
10. $6x^2+11x-10$. 11. $2x^2-3xy-5y^2$. 12. $3x^2+14xy-5y^2$.
13. $6x^4+29x^2-5$. 14. $7x^6+19x^3-6$. 15. $10x^2-29x-21$.
16. $8x^2y^2+6xy-5$. 17. $7x^4-19x^2y-6y^2$. 18. $12x^2-4x-21$.
19. $(x-1)(3x+2)$. 20. $(x+1)(3x-2)$. 21. $(x+2)(3x-1)$.
22. $(x-5)(2x+1)$. 23. $(x+1)(2x-5)$. 24. $(x-3y)(3x+2y)$.
25. $(x+6y)(3x-y)$. 26. $(x-3)(2x+1)$. 27. $(x+1)(2x-3)$.
28. $(x+2)(3x-4)$. 29. $(x+1)(3x-8)$. 30. $(x+2)(5x-3)$.
31. $(x-6)(5x+1)$. 32. $(3x+2y)(5x-y)$.
33. $(x-y)(15x+2y)$. 34. $(x+7)(5x-2)$.
35. $(x^2+7)(7x^2-1)$. 36. $(2x^3+1)(3x^3-10)$.
37. $(x+2)(5x-6)$. 38. $(x-1)(5x+12)$.
39. $(2x-1)(3x+7)$. 40. $(x-2)(14x+1)$.
41. $(x-3)(5x+7)$. 42. $(x-21)(5x+1)$.
43. $(x+6y)(3x-2y)$. 44. $(x+12)(3x-1)$.
45. $(4x-5)(5x+4)$. 46. $(x-12)(12x+1)$.
47. $(3x^2+7y)(5x^2-y)$. 48. $3(x+1)(2x-5)$.
49. $(2x-5)(3x+4)$. 50. $(2x+5)(3x-7)$.
51. $(4x+5)(6x-5)$. 52. $(4x-7)(3x+2)$.
53. $(2x-3)(5x+7)$. 54. $(2x+7)(4x-5)$.
55. $(x+10)(4x-3)$. 56. $(4x-1)(8x+3)$.
57. $(7x+8)(2x-7)$. 58. $(7x+2)(14x-3)$.
59. $(9x+4)(18x-5)$. 60. $(5x-6)(10x+7)$.
61. $8(2x+y)(3x-2y)$. 62. $(6x+35y)(x-7y)$.

EXERCISE XVIII.

p. 74.

1. (i) $4x^2-1=(2x+1)(2x-1)$;
(ii) $25x^2-9y^2=(5x+3y)(5x-3y)$;
(iii) $x^4-y^4=(x^2+y^2)(x^2-y^2)$;
(iv) $x^6-y^6=(x^3+y^3)(x^3-y^3)$;
(v) $(x+y)^2-16z^2=(x+y+4z)(x+y-4z)$;
(vi) $x^2-(y-z)^2=(x+y-z)(x-y+z)$.
2. $(x+1)(x-1)$. 3. $(7+x)(7-x)$.
4. $(10+ab)(10-ab)$. 5. $(6ab+5cd)(6ab-5cd)$.
6. $(9x^4+ay^3)(9x^4-ay^3)$. 7. $(2ax^2+3b^2y)(2ax^2-3b^2y)$.
8. $(4x^3+y^4)(4x^3-y^4)$. 9. $(11x+8a^2)(11x-8a^2)$.

10. $(x^3+6)(x^3-6)$.
11. $(5x^5+3)(5x^5-3)$.
12. $(x^2+4y^8)(x+2y^4)(x-2y^4)$.
13. $(ab+7x^2)(ab-7x^2)$.
14. $(a+b+c)(a+b-c)$.
15. $(a+b+c)(a-b-c)$.
16. $(a-b+c)(a-b-c)$.
17. $(a+b-c)(a-b+c)$.
18. $(a-b+x-y)(a-b-x+y)$.
19. $(a+b+x+y)(a+b-x-y)$.
20. $(3a+b+c)(3a-b-c)$.
21. $(a+3b+3c)(a-3b-3c)$.
22. $(5a+2b-2c)(5a-2b+2c)$.
23. $(6a+7b-7c)(6a-7b+7c)$.
24. $(x-y+1)(x-y-1)$.
25. $(x+2y+2z)(x+2y-2z)$.
26. $(2z+x-2y)(2z-x+2y)$.
27. $2(2x+1)(2x-1)$.
28. $3x(x+3)(x-3)$.
29. $2(x^2+yz)(x^2-yz)$.
30. $y^3(5x^2+y^2)(5x^2-y^2)$.
31. $x^3z(12y+x)(12y-x)$.
32. $ab(a+b)(a-b)$.
33. $a^3b^3(a+b)(a-b)$.
34. $13x^3(2x+1)(2x-1)$.
35. $(a^2+b^2)(a+b)(a-b)$.
36. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
37. $(4a^2+9b^2)(2a+3b)(2a-3b)$.
38. $a^4(a^2+25b^2)(a+5b)(a-5b)$.
39. $2x(4x^2+1)(2x+1)(2x-1)$.
40. $x^3(x^2+1)(x+1)(x-1)$.
41. $ab(a^2+9b^2)(a+3b)(a-3b)$.
42. $3(x+y-z)(x-y+z)$.

EXERCISE XIX.

p. 75.

1. $(x+y+4)(x+y-4)$.
2. $(x+2y-1)(x-2y-1)$.
3. $(x-2y+z)(x-2y-z)$.
4. $(x-5y+3z)(x-5y-3z)$.
5. $(x+3y+7)(x-3y+7)$.
6. $(x-6y+5z)(x-6y-5z)$.
7. $(2x+3y+z)(2x+3y-z)$.
8. $(3x-5y+2z)(3x-5y-2z)$.
9. $(2x+y+7)(2x-y+7)$.
10. $(x-y+z)(x-y-z)$.

p. 76.

11. $(x+y+z)(x+y-z)$.
12. $(2x-5y+z)(2x-5y-z)$.
13. $(x+y-z)(x-y+z)$.
14. $(x+y+1)(x-y-1)$.
15. $(2x+3y-1)(2x-3y+1)$.
16. $(3x+5y+z)(3x-5y-z)$.
17. $(4x+3y-2z)(4x-3y+2z)$.
18. $(7x+3y+7)(7x-3y-7)$.
19. $(x+5y-3z)(x-5y+3z)$.
20. $(a+b+c+d)(a+b-c-d)$.
21. $(a+b+c-d)(a-b+c+d)$.
22. $(5x-1)(x+5)$.
23. $(3x-1)(x+5)$.
24. $(11x+3)(x+15)$.
25. $(17x-16)(13x-4)$.
26. $(11x-14)(7x-22)$.
27. $(x^2+x+1)(x^2-x+1)$.
28. $(x^2+2xy+4y^2)(x^2-2xy+4y^2)$.
29. $(x^2+3x+1)(x^2-3x+1)$.
30. $(x^2+3xy-y^2)(x^2-3xy-y^2)$.
31. $(x^2+xy+2y^2)(x^2-xy+2y^2)$.

p. 76. 32. $(x^2+2x+2)(x^2-2x+2)$.
33. $(x^2+4xy+8y^2)(x^2-4xy+8y^2)$.
34. $5(x+1)(5x-7)(x^2+1)$. 35. $(x+1)(x+2)(x+3)(x+4)$.
36. $4(x-1)(x+1)(x-2)(x+3)$.
37. $12(x+2)(2x+1)(2x+3)(3x-1)$.
38. $24(x-2)(x-5)(2x+1)(3x+4)$.

EXERCISE XX.

p. 77. 1. $(x+1)(x^2-x+1)$. 2. $(x-1)(x^2+x+1)$.
3. $(1+2x)(1-2x+4x^2)$. 4. $(3x-1)(9x^2+3x+1)$.
5. $(3x-y)(9x^2+3xy+y^2)$. 6. $(x+4y)(x^2-4xy+16y^2)$.
7. $(6x-y)(36x^2+6xy+y^2)$. 8. $(3x-2y)(9x^2+6xy+4y^2)$.
9. $(7x+1)(49x^2-7x+1)$. 10. $(xy+z)(x^2y^2-xyz+z^2)$.
11. $(xy-10)(x^2y^2+10xy+100)$. 12. $(8+x)(64-8x+x^2)$.
13. $(xy-2z)(x^2y^2+2xyz+4z^2)$. 14. $(7x-2y)(49x^2+14xy+4y^2)$.
15. $(2x+5y)(4x^2-10xy+25y^2)$. 16. $(1+10x)(1-10x+100x^2)$.
17. $2x(2x+1)(4x^2-2x+1)$. 18. $xy(x-y)(x^2+xy+y^2)$.
19. $x^2y^2(x+2y)(x^2-2xy+4y^2)$. 20. $5x(2+5x)(4-10x+25x^2)$.
21. $2(xyz-2)(x^2y^2z^2+2xyz+4)$. 22. $(x^2+3y)(x^4-3x^2y+9y^2)$.
23. $9x^6(x+2)(x^2-2x+4)$. 24. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
25. $(x^2+1)(x^4-x^2+1)$.
26. $(x+1)(x-1)(x^2+x+1)(x^2-x+1)$.
27. $(x^2+4)(x^4-4x^2+16)$.
28. $(x+2)(x-2)(x^2+2x+4)(x^2-2x+4)$.
29. $(x^2+9y^2)(x^4-9x^2y^2+81y^4)$.
30. $(x+3y)(x-3y)(x^2+3xy+9y^2)(x^2-3xy+9y^2)$.
31. $9(x+y)(x^2+xy+y^2)$. 32. $2(2x+y)(31x^2+40xy+13y^2)$.
33. $2(2x-y)(31x^2-40xy+13y^2)$. 34. $y(27x^2-9xy+y^2)$.

EXERCISE XXI.

p. 80. 1. 5. 2. 10. 3. 7. 4. 12. 5. 19. 6. 2.
7. 7. 8. 2. 9. 1. 10. 4. 11. 1. 12. 2.
13. 8. 14. 5. 15. 2. 16. 4. 17. 6.
18. 10. 19. 3. 20. 1. 21. 5. 22. 5.
23. $4a+3b$. 24. $5a$. 25. $2b$. 26. a. 27. $3a+2$.
28. $a-b$. 29. $a+b$. 30. $a-b$. 31. $a+b$. 32. $a-b$.
33. $a-b$. 34. $a+b+1$. 35. $a+b$. 36. $a+4$. 37. $a-b$.

p. 81. 38. a^2+ab+b^2. 39. a^2-3a+9. 40. $a+b$. 41. $b-a$.

EXERCISE XXII.

p. 82. **1.** $n(x+y-2z)$ shillings.
2. (i) $s=n(x+y-2z)$; (ii) 20 days. **3.** 67, 27. **4.** 24, 25.
5. 88. **6.** £2, £3. **7.** 36. **8.** 10. **9.** 8, 15. **10.** 81.

p. 83. **11.** A has £150, B has £250. **12.** 31, 7. **13.** 99.
14. 8. **15.** 36. **16.** 37 sovereigns, 92 shillings.
17. A, 23 ; B, 17. **18.** 2 ; £6. 10*s*. **19.** 4, 5, 6.
20. 4 sovereigns ; 16 half-crowns ; 6 shillings.
21. A, £80 ; B, £90 ; C, £230. **22.** 300.

p. 84. **23.** 5 engravings, 10 books. **24.** 22 ft.
25. 27 miles from the starting-point of the slower coaches.
26. 636. **27.** 14, 28. **28.** 12, 24, 36, 48.
29. 15, 26, 37, 48, 59. **36.** $5n-2$; 41. **37.** 3.
38. 7 ; 72. **39.** 5.

EXERCISE XXIII.

p. 89. **1.** 14, 7. **2.** 10, 2. **3.** 6, 2. **4.** 12, 3.
5. 2, 8. **6.** 10, 4. **7.** 3, 4. **8.** 6, 21.
9. 10, 6. **10.** 2, 8. **11.** 6, 2. **12.** 9, 3.
13. 3, 2 ; 3, 2, 5. **14.** $11y=3z$; $11x=5z$; 5, 3, 11.
15. $4x+9=a$; 5, 13. **16.** 41, 44, 47. **17.** $x+a=8$; 5, 11.
18. $x=4a-3b$, $y=3a-2b$; $a=8$, $b=9$. **19.** 40.

EXERCISE XXIV.

p. 92. **1.** 80, 32. **2.** 9, 7. **3.** 10, 4. **4.** 9, 8. **5.** 2, 3.
6. 7, 4. **7.** 41, 35. **8.** 10, 4. **9.** 9, 13. **10.** 9, 15.
11. 10, 3. **12.** 5, 2. **13.** 2, 3.
16. $x+2y$ cannot be equal to both 3 and 2 *at the same time*.
17. 1, 4 ; 4, 3 ; 7, 2 ; 10, 1. **18.** 2, 2. **19.** 5, 3.
20. 4, 3. **21.** 11, 7. **22.** $a+b$, $a-b$. **23.** $a+b$, $a+2b$.
24. $a+b$, $a-b$. **25.** $2a+b-c$, $a+2b-2c$. **26.** $a-1$, a.
27. $a+b$, $a+b$. **28.** b, a. **29.** $a-1$, a^2.
30. b, a. **31.** a^2+a+1, a^3. **32.** $a+b$, a.
33. $2a+1$, $2b-1$. **34.** a^3+1, a^3-1. **35.** $a+b$, $a-b$.

EXERCISE XXV.

p. 95. **1.** 28. **2.** 259. **3.** $a-5$. **4.** $x+8y$. **5.** 17, 2.
p. 96. **6.** 3 shillings. **7.** 28 crowns 92 shillings.

p. 96. **8.** 25*s*. 4*d*., 12*s*. **9.** 18 ft., 12 ft.
10. Tea, 2*s*. 4*d*.; coffee, 1*s*. 8*d*. **11.** 21. **12.** 59.
13. 48, 39. **14.** 36 yds. at 3*s*. 6*d*.; 50 yds. at 4*s*. 6*d*.
15. 42 lbs. at 2*s*. 4*d*.; 48 lbs. at 1*s*. 10*d*.
16. *A* has 28 shillings and 7 pence; *B* has 12 shillings and 4 pence.

p. 97. **17.** Cloth, £23; silk, £17.
18. Widow, £7250; son, £3100; daughters, £1550 each.
19. 1800 tons. **20.** $2x=a-b$; $a=43$, $b=23$.
21. $2x=b-a-4$; $b=94$. **22.** 300 acres. **24.** 720 miles.

p. 98. **25.** 68 inches, 86 inches.
26. Small type, 11 pages; large type, 13 pages.
27. *A*, 17 miles; *B*, 12 miles. **28.** £2. 5*s*. **29.** 21 lbs., 14 lbs.
30. *A*, 5*s*. 10*d*.; *B*, 6*s*. 3*d*. (five shillings and 15 pennies).
31. £158. 8*s*.

p. 99. **32.** *X*, 32 m.p.h.; *Y*, 21 m.p.h. **33.** *A*, 28; *B*, 21.
34. 20 ft., 15 ft. **35.** 3000 men.
36. (ii) 1, 24; 2, 12; 3, 8; 4, 6: (iii) 3, 8. **37.** $x=6$, $y=5$.
38. 24 ft. long, 18 ft. wide. **39.** 25 chairs.

EXERCISE XXVI.

p. 102. **1.** $x=11$, $y=12$, $z=13$. **2.** $x=6$, $y=8$, $z=9$.
3. $x=3$, $y=6$, $z=1$.

p. 103. **4.** $x=10$, $y=5$, $z=1$. **5.** $x=4$, $y=5$, $z=7$.
6. $x=3$, $y=4$, $z=5$. **7.** $x=5$, $y=3$, $z=1$.
8. $x=5$, $y=6$, $z=7$. **9.** $x=11$, $y=13$, $z=17$.
10. $x=5$, $y=5$, $z=5$. **11.** $x=4$, $y=5$, $z=3$. **12.** 5.
13. The equations are inconsistent, for $y-z$ cannot stand for both 5 and 10.
14. (ii) 5, 2; 7, 7; 9, 12; ...: (iii) 5, 2, 18; 7, 7, 11; 9, 12, 4.
15. (ii) 8, 1; 12, 4; 16, 7: (iii) 8, 1, 7; 12, 4, 14; 16, 7, 21: (iv) the number is unlimited.

p. 104. **17.** *A*, £10; *B*, £12; *C*, £18. **18.** 7 miles.
19. 660 yds. **20.** 473. **21.** $a=7$, $b=6$, $c=5$.
22. $a=5$, $b=4$, $c=3$. **23.** $a=b=1$; $c=2$.
24. 25*s*., 15*s*., 5*s*. **25.** Sheep, 45*s*; lamb, 17*s*.; pig, 34*s*.

EXERCISE XXVII.

p. 110. 1. a. 2. $\frac{xy^3}{2}$. 3. $3y^2$. 4. b.

5. $\frac{2y^3}{3x^3}$. 6. $\frac{5c^3}{3a^2b^2}$. 7. $\frac{2a^2b^3}{3c^3}$. 8. $\frac{2(b+c)}{a}$.

9. $\frac{xy^2(x+y)}{x-y}$. 10. $\frac{x(2x+y)}{2(x+y)}$. 11. $\frac{ad}{3}$. 12. $2x$.

13. $\frac{4yz^3}{x^2}$. 14. $\frac{3b}{a+b}$. 15. $\frac{4ab}{b^2-a}$. 16. x^2y+xy^2+2.

17. $(x-1)^2$. 18. ab. 19. ab. 20. x^3-1.

21. $2x^2-1$. 22. $x-4$. 23. $x-y$. 24. $x+y+z$.

25. (i) $ad+bc$; (ii) $6x+4y-3z$; (iii) $2bcx+3acy-4abz$; (iv) $5x^4+6y^4$; (v) $12x-\frac{25y}{3}$.

p. 111. 26. (i) $\frac{d^2}{b^2}$; (ii) $\frac{c^4}{b^2d^2}$; (iii) $\frac{x^2z^2}{y^4}$; (iv) $\frac{x^4}{y^2\omega^2}$; (v) $\frac{l^4p^2}{m^2n^4}$; (vi) $\frac{l^2p^4}{m^4n^2}$.

27. (i) $x-1$; (ii) $\frac{3(x+1)}{2}$; (iii) $\frac{2x+1}{2}$; (iv) $x-1$; (v) 2; (vi) $\frac{y-z+x}{y+z+x}$; (vii) $ab(a+b)$; (viii) $3a-2b$; (ix) $\frac{a^2b^2}{(a-b)^2}$; (x) $(3y-2x)$.

28. (i) $3(a+b)$; (ii) $2a-b$; (iii) $3(a+b)^2$; (iv) 2; (v) $2b$.

29. (i) $1+\frac{6}{x}$; (ii) $5x+\frac{4}{x}$; (iii) $x+1-\frac{10}{x+1}$.

p. 112. 30. (i) $2b$; (ii) $2ab$; (iii) $16x^2-12xy+9y^2$; (iv) $2x-1$; (v) $\frac{b(a+b)}{a}$.

31. (i) $\frac{2x-3y}{x-2y}$; (ii) $\frac{2x-3}{x-2}$.

EXERCISE XXVIII.

p. 115. 1. $\frac{x+6}{x}$. 2. $\frac{ac-b}{c}$. 3. $\frac{2xy+6}{y}$. 4. $\frac{2x^3+4}{x}$.

5. $\frac{3x^5+12}{x^2}$. 6. $\frac{acy-by+cx}{cy}$. 7. $\frac{x^3-3x+9}{x^2}$.

8. $\frac{4+2x+x^2}{x}$. 9. $\frac{6x+4y-3z}{12a}$. 10. $\frac{29x}{a}$.

11. $\frac{a^2-5}{a-1}$. 12. $\frac{a^2}{a-3b}$. 13. $\frac{5x-1}{6}$. 14. $\frac{a+17b}{12}$.

p. 115. 15. $\frac{22a-3b}{12}$. 16. $\frac{19y-2x}{15}$. 17. $\frac{59y-69x}{14}$. 18. $\frac{24x+25y}{12}$.

19. $\frac{5x-7}{12}$. 20. $\frac{28b-27c}{24}$. 21. $\frac{a}{4}$. 22. $\frac{x}{5}$.

23. $\frac{25a}{12(x-1)}$. 24. $\frac{a}{4x+8y}$. 25. $\frac{5a}{4x+4}$. 26. $\frac{x^3}{(x-3y)^2}$.

27. $\frac{a^2}{a+b}$. 28. $\frac{a^2}{a-b}$. 29. $\frac{b^2}{2a+b}$. 30. $\frac{4b^2}{3a-2b}$.

p. 116. 31. $\frac{6a+15}{(a+1)^2}$. 32. $\frac{12a^2-16a}{a^3-2a^2+a}\left[=\frac{4(3a-4)}{(a-1)^2}\right]$.

33. $\frac{x^3}{x^2-xy+y^2}$. 34. $\frac{2x^3}{x-y}$. 35. $\frac{b^3}{4a^2-2ab+b^2}$.

36. $\frac{b^2(b-a)}{a+b}$. 37. $\frac{a^3}{a-b}$. 38. $2a^2+2b^2-4ab[=2(a-b)^2]$.

39. $\frac{8a+2b}{b}$. 40. $xy(x+y)$. 41. $x(x-y)$.

42. $x+y$. 43. $9x^2+4$. 44. $\frac{9z}{x}$. 45. $x+y$.

46. $\frac{(3a+2b)(4a+3b)-(2a+3b)(3a+4b)}{3(a+b)}=2(a-b)$.

EXERCISE XXIX.

p. 119. 1. $x=\frac{ab}{c}$. 2. $x=\frac{3c}{8}$. 3. $x=\frac{5a}{9}$. 4. $x=\frac{7a}{6}$.

5. $x=\frac{a(b-c)}{c}$. 6. $x=\frac{a(c-b)}{c}$. 7. $x=\frac{12a}{6-a}$. 8. $x=\frac{ab}{c}$.

9. 18. 10. 4. 11. 33. 12. 19. 13. 7.
14. 5. 15. 1. 16. 13. 17. 12. 18. 11.
19. 8. 20. 13. 21. 5.

p. 120. 22. 32. 23. 4. 24. 2. 25. 4. 26. 5.
27. 24. 28. 16. 29. 3, 8. 30. 20, 30. 31. 9, 5.
32. 2, 14. 33. 1, 3. 34. 9, 7. 35. 10, 4. 36. 81, 16.
37. 11, 9. 38. 7, 9. 39. 7, 30.

p. 121. 40. $\frac{b-c}{2a}$. 41. $a+b$. 42. $b-a$. 43. $a+3b$. 44. $\frac{ab}{b+1}$.

45. $a-b$. 46. $2a-b$. 47. $a+b$. 48. $ab-a-b$. 49. $a+b$.

50. $a+b+c$. 51. $\frac{r}{p+q}$, $\frac{r}{p+q}$. 52. $(a+1)^2$, $(a-1)^2$.

53. $a-c$, $2(b-c)$. 54. $\frac{1}{6}(a^2-3ab+b^2)$, $\frac{1}{6}(a^2+3ab+b^2)$.

EXERCISE XXX.

p. 122. **1.** (i) $5x$ pence ; (ii) $\frac{xy}{12}$ pence.

2. (i) $\frac{240x}{y}$ apples ; (ii) $\frac{xz}{12y}$ shillings.

3. (i) $\frac{x}{3}$ hours ; (ii) $\frac{x}{y}$ hours.

p. 123. **4.** (i) $\frac{y}{x}$ m.p.h. ; (ii) $\frac{1760y}{3x}$ yds. **5.** $\frac{88x}{3}$ yds. per min.

6. $\frac{45y}{22m}$ secs. **7.** $\frac{5xz}{3y}$ dwt. **8.** $\frac{60y}{x}$ days.

9. $\frac{144xy}{ab}$. **10.** $\frac{144x^2}{an}$. **11.** $\frac{kabc}{144pq}$ feet.

12. (i) xy hours ; (ii) $\frac{xy}{z}$ hours. **13.** $3x=2y$.

14. (i) $\frac{lb}{18}$ yards ; (ii) $\frac{lbz}{18}$ shillings.

15. (i) $\frac{2h(b+l)}{9}$ sq. yds. ; (ii) $\frac{2h(b+l)}{3c}$, $\frac{h(b+l)z}{216}$ shillings.

16. $\frac{bpq}{ac}$ acres.

p. 124. **17.** $x=2y$. **18.** £12. **19.** £24. **20.** 24 feet.
21. 68, 57. **22.** 107. **23.** 25, 30, 45.
24. 150 acres. **25.** 654. **26.** £23.
27. £50. **28.** 24, 18. **29.** £450, £400.

p. 125. **30.** 217, 403. **31.** 96. **32.** 17 shillings.
33. 60 shillings, 50 francs. **34.** £4. 12*s*., £2. 8*s*.
35. 5 miles uphill, 6 miles on level, 7 miles downhill.
36. 1400. **37.** 330. **38.** 1920.

EXERCISE XXXI.

p. 129. **1.** x^2. **2.** $3x^3$. **3.** x^2. **4.** $5x^5$. **5.** xy^2z^4.
6. xy^2z^8. **7.** x^2y^6. **8.** $8x^{32}$. **9.** $3x^9$. **10.** xy^3.
11. nx^n. **12.** $x^3y^{n^2}$. **13.** x^{n+1}. **14.** x^2. **15.** x.
16. 14. **17.** 21. **18.** 22. **19.** 104. **20.** 12.
21. 15. **22.** 28. **23.** $2a^2$. **24.** $5a^5$. **25.** 7.
26. 4. **27.** $4a^2$. **28.** 30. **29.** 84. **30.** 38.
31. 6. **32.** 4. **33.** 4.
34. (i) 2 ; (ii) 3 ; (iii) 3. **35.** (i) 40 ; (ii) 4 ; (iii) 25.
36. (i) 336 sq. in. ; (ii) 144 sq. in. ; (iii) 840 sq. in. ;
(iv) 1008 sq. in. **37.** 3696 sq. in.

EXERCISE XXXII.

p. 136. 1. (i) Parallelogram; (ii) rectangle; (iii) rhombus; (iv) square; (v) square.

2. (i) 40; (ii) 28. 3. (i) 16; (ii) 12; (iii) 21; (iv) 9.

4. (i) 30; (ii) 20; (iii) 45; (iv) 30.

5. (i) 31; (ii) 17; (iii) 22.

6. (i) 18; (ii) 13; (iii) 17; (iv) 24.

7. (i) 44; (ii) 72; (iii) 138; (iv) 58; (v) 80; (vi) 83; (vii) 78. 8. 15, 26, 34, 50.

p. 137. 9. (i) 5; (ii) 10; (iii) 13; (iv) 17. 10. 65.

11. 5. 12. $x^2+y^2=r^2$. 13. $(x-a)^2+(y-b)^2=r^2$.

14. If O is the origin, P the point [24, 7], Q the point [12, 16], $OP=25$, $OQ=20$, $PQ=15$.

15. The squares on the sides are respectively 100, 125, 25.

EXERCISE XXXIII.

p. 142. 1 (*c*). (i) [6, 6], [15, 15]; (ii) [6, 9], [10, 15]; (iii) [6, 13], [4, 15]; (iv) [6, 9], [12, 15]; (v) [6, 15], [6, 15]; (vi) [6, 4], [28, 15]; (vii) [6, 13], [2, 15]; (viii) [6, 12], [8, 15]; (ix) [6, 12], [6, 15].

2. $x=6$. 3. $y=8$. 4. $x=y$. 5. $x=3y$.

6. $x+y=15$. 7. Rectangle; 16 units of area.

8. [7, 9]. 9. [10, 9]. 10. [12, 16]. 11. [20, 12].

12. [10, 11]. 13. [12, 13]. 14. [13, 15]. 15. [12, 10].

p. 143. 16. 11. 17. 4. 18. 2.

EXERCISE XXXIV.

p. 146. 2. (i) 10·1 ft.; (ii) between 70 yds. from firing point and 760 yds. from firing point, *i.e.* a distance of 690 yds.

3. 10·1 ft.; 43·7 ft. 6. (i) 9·2, 6·7, 5·8; (ii) 13·3, 30.

7. 0·67, 0·73, 0·87, 0·93 ft.

p. 147. 8. (i) 1350, 450; (ii) 2·7, 7·5, 3·75. 9. 7·5, 11·7.

10. (i) E too high when $C=7, 2, 0·5, 0·2$; (ii) E too low when $C=3$; (iii) E correct when $C=5, 1·5, 1$.

EXERCISE XXXV.

p. 152. 1. -3. 2. -13. 3. $-15a$. 4. $-4a$.

5. $4a$. 6. 0. 7. $-4a+6$. 8. $2a$.

9. $a-1$. 10. $-2a$. 11. $3b-5a$. 12. 0.

13. $2x$. 14. $-12x$. 15. $-2a$. 16. $-5x$. 17. $2x$.

p. 153. 18. 5. 19. -5. 20. 12. 21. $2+n$.
22. -8. 23. $2-n$. 24. 0. 25. $2n$.
26. $x+4$. 27. $3+4x$. 28. $x-10$. 29. $-4x-3$.
30. 4. 31. -14. 32. 13. 33. -23.
34. -3. 35. $b-2a$. 36. 1. 37. 10.
38. -20. 39. -28. 40. 0. 41. -20.
47. $-x+8$, $-x+18$. 48. $a+5$, $-a-2b-5$.
49. $-a-14d$, $-a-6d$.

EXERCISE XXXVI.

p. 155. 1. (i) £$(a-b)$; (ii) £$(b-a)$. 2. (i) £$(a-b)$; (ii) £$(b-a)$.
3. (i) £$(-6x)$; (ii) £$(6x)$. 4. (i) £$(-2x)$; (ii) £$2x$.
5. 98 degrees.
6. (i) A is x miles *south* of B; (ii) A is two metres *below* sea-level; (iii) a train is moving *northwards* at the rate of 30 miles an hour; (iv) B *owes* A £10; (v) A is placed *x yds. behind scratch.*
7. A, $(a+x)$ yds.; B, $(a-y)$ yds.
8. $(a-b-c+d)$ ft.; $(-a+b+c-d)$ ft.
9. -3, 15, -18, -6.
10. (i) $-10a+5b+4c-11d$ feet; (ii) $10a-5b-4c+11d$ feet; (iii) 14; (iv) $a=-22$, *i.e.* road *falls* a feet per mile for first 10 miles.

p. 156. 11. (i) $-14+10=-4$; (ii) $-12+20=8$; (iii) $+5+10=15$; (iv) $5-20=-15$. 12. (-2) ounces.
13. (i) $v=-24$ (*i.e.* stone is moving with a velocity of 24 ft./sec. *downwards*), $s=16$; (ii) $v=-56$ (velocity of 56 ft./sec. *downwards*), $s=-24$ (stone is 24 ft. *below* point of projection); (iii) $v=32$, $s=384$; (iv) $v=0$ (stone at its highest point, momentarily at rest), $s=400$; (v) $v=0$, $s=1176$; (vi) $v=-160$ (velocity of 160 ft.-sec. *downwards*), $s=0$ (stone has returned to point of projection).
15. (i) 2 seconds; (ii) after 3 seconds, at a height of 96 ft. above point of projection; (iii) 5 seconds.

EXERCISE XXXVII.

p. 160. 1. 2. 2. 8. 3. -8. 4. -2. 5. -15.
6. $+15$. (Note that $-5(-3)$ is not the same expression as $(-5)(-3)$, which at present is meaningless.)

p. 160. 7. 2. 8. 4. 9. -8. 10. 0. 11. -12.
12. 0. 13. -8. 14. 2. 15. 0. 16. 6.
17. 8. 18. $-3a$. 19. $11a$. 20. $7a$. 21. $c-b$.
22. $2(c-b)$. 23. $2c$. 24. $2a+b+c$. 25. $a+2b-c$.
26. $a-b-4c$. 27. $-b-c$. 28. $-2c$. 29. $2y$. 30. $-2y$.
31. 7. 32. -7. 33. $7a$. 34. a. 35. $a+4$.
36. $a-4$. 37. b. 38. $4a-b$. 39. 11. 40. 3.
41. 15. 42. -15. 43. 6. 44. -1. 45. 1.
46. 1. 47. 1, 2. 48. 12. 49. 5. 50. -1.

EXERCISE XXXVIII.

p. 163. 7. 4. 8. $8a$. 9. $3a-6b+3c$. 10. $-a$.
11. (i) $-2a-7b+13c$; (ii) $a+4b+3c$. 12. $4x-3y-5z$.
13. $4x+y-z$. 14. $2x-2y+3z$. 15. $-a-b+c-d$.
16. 0. 17. $-a+b+2c$. 18. $-2c$.
19. $2a-2c$. 20. $-4x+2a$. 21. $a+b+c$.
22. $-23x+5y+28z$. 23. $2ca-2x$. 24. $-4x+10y-z$.
25. $-4a-4b$, $c+4a$. 26. $-x-z$; $x+y-z$.

p. 164. 27. $x^2-3ax+a^2$, ax.
28. $2a^3+2a^2b+2ab^2+2b^3$, $2a^3+4ab^2+2b^3$.
29. $6x^3+9x^2+6x-15$. 30. $17a+9b-26c$.
31. $6ab$. 32. 0. 33. $-7b$. 34. $7a$.
35. $bx-by$. 36. $-3a$. 37. $3c$. 38. $4a+6c$.
39. $3b^2-2c^2$. 40. $73x-24$. 41. $11x-9y+11z$.

EXERCISE XXXIX.

p. 165. 2. X, $\frac{1}{2}(b-a)$; Y, $\frac{1}{3}(2b-a)$; Z, $\frac{1}{4}(3b-a)$; X, 150 ft. above; Y, 100 ft. below; Z, 225 ft. below.

p. 166. 3. $2x=a+b+c$. 4. (i) 4; (ii) 0; (iii) 2; (iv) 0.
5. $\frac{1}{2}(3b-a)$; A's age *will be* three times B's age *in* 5 *years' time*.
6. $x=a-b$. 7. 12.
8. $y=x+a+b$; A will be born in 3 years' time.
9. $x=b-a$; B pays $(a-b)$ shillings to A.
10. $x=\frac{1}{3}(3c+2b-a)$; B pays £50 to A.
11. $x=\frac{1}{3}(a+c-2b)$, $y=\frac{1}{3}(a+b-2c)$; A pays £20 to B and A pays £170 to C.
12. $x=\frac{1}{4}(a+b-3c)$, $y=\frac{1}{2}(a-b-c)$; B pays £150 to C and A pays £200 to B.

EXERCISE XL.

p. 171. 1. (i) 28 ; (ii) 27 ; (iii) 51 ; (iv) 78.

2. (i) 56 ; (ii) 78 ; (iii) 50 ; (iv) 120.

3. Two sides of a right-angled triangle are greater than the hypotenuse.

4. (i) 10 ; (ii) 13 ; (iii) 17 ; (iv) 29.

6. (i) 25 ; (ii) 20 ; (iii) 37.

7. (i) [5, 5] ; (ii) [5, −4] ; (iii) [−2, 0] ; (iv) [−1, −4].

8. (i) [16, 0], [0, 15] ; (ii) [12, 0], [0, −9] ; (iii) [12, 0], [0, 8] ; (iv) [−21, 0], [0, −12].

p. 172. 10. 5, 4. **11.** −8, 7. **12.** −28, 20. **13.** −8, −6. **15.** 4.

17. This is the locus of points which are equidistant from (a, b), (a', b') ; its equation is therefore $(x-a)^2+(y-b)^2 = (x-a')^2+(y-b')^2$. This reduces to

$$2x(a-a')+2y(b-b')=a^2-a'^2+b^2-b'^2.$$

EXERCISE XLI.

p. 177. 1. $-xyz$. **2.** $9x^2$. **3.** $-8x^3$. **4.** $8x^3$. **5.** x^3.

6. $-x^4$. **7.** −1. **8.** 1. **9.** 17. **10.** 0.

11. 6. **12.** 0. **13.** (i) 1 ; (ii) −1 ; (iii) 1.

14. (i) 8 ; (ii) 72 ; (iii) −180 ; (iv) 36000 ; (v) 250000 ; (vi) −7 ; (vii) 14 ; (viii) 0 ; (ix) 0 ; (x) 0.

EXERCISE XLII.

p. 180. 1. $-\frac{a}{b}$. **2.** $-a$. **3.** $a-b$. **4.** $\frac{x+2}{x-2}$. **5.** $x-y$.

6. a. **7.** $-b$. **8.** $-\frac{b^2}{a-b}$. **9.** $-b(a^2+ab+b^2)$.

10. $\frac{a+b}{a-b}$. **11.** $\frac{a+2b}{a-b}$. **12.** $2b$. **13.** $-2ab$.

14. $-\frac{c}{b}$. **15.** $-\frac{b}{c}$. **16.** $x-4$. **17.** $4-x$.

18. $x(y-2x)$. **19.** $3y-2x$. **20.** $-3x-4y$. **21.** 2.

22. 125. **23.** −25. **24.** 3. **25.** −9.

26. 0. **27.** −3. **28.** 31. **29.** 2.

EXERCISE XLIII.

p. 182. 1. $-2a$. **2.** $3a^2$. **3.** $-2a^3$. **4.** a^2.

5. $-a^2$. **6.** −28. **7.** −18. **8.** −84.

p. 182. 9. -105. 10. $\frac{x^3y^4}{a^5}$. 11. $\frac{x^2y^3}{a^5}$. 12. $\frac{x^ny^{n+1}}{a+b}$.
13. $1-x$. 14. $a+b$ and $-a-b$.

p. 183. 15. $2x-3$. 16. $5x+8y$. 17. $4x^2-7$. 18. $2x^3-9yz$.
19. $(x-1)(x-2)(x-3)$. 20. $(x+1)(x-3)(3x-1)$.
21. $(x-1)(2x-3)(5x+6)$. 22. $2x-y+52$.
23. $3x+2y+4z$. 24. $x-2y+1$. 25. $3x^2+5y^2-2$.
26. $xy+3y+2$. 27. $+a, -a$. 28. $-a+b, -a-b$.
29. $5, 0$. 30. $6, -1$. 31. $-2, -4$.
32. $+2, -2$. 33. $+3, -3$. 34. $+1, -1$.
35. -2. 36. $-a-3b$. 37. $-a$. 39. -1.

EXERCISE XLIV.

p. 185. 1. (i) $a^2+b^2+c^2-2ab+2ac-2bc$;
(ii) $a^2+b^2+c^2-2ab-2ac+2bc$;
(iii) $4a^2+b^2+9c^2-4ab-12ac+6bc$;
(iv) $9a^2+16b^2+25c^2+24ab-30ac-40bc$;
(v) $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$;
(vi) $a^2+b^2+c^2+d^2-2ab-2ac-2ad+2bc+2bd+2cd$;
(vii) $a^2+4b^2+25c^2+d^2-4ab-10ac+2ad+20bc-4bd-10cd$;
(viii) $a^2+4b^2+9c^2+16d^2-4ab-6ac+8ad+12bc-16bd-24cd$.

p. 186. 2. (i) $x^4-4x^3+2x^2+4x+1$;
(ii) $x^6-6x^4-8x^3+9x^2+24x+16$;
(iii) $x^6-4x^5-2x^4+20x^3-7x^2-24x+16$;
(iv) $x^6-2x^5-9x^4+4x^3+31x^2+30x+9$;
(v) $x^4+2ax^3+(a^2-2b)x^2-2abx+b^2$;
(vi) $x^6-2ax^5+a^2x^4-2bx^3+2abx^2+b^2$.
3. (i) 0; (ii) $4(x^2+y^2+z^2)$.
4. (i) $x+y$; (ii) $2(y-x)$; (iii) $2(b-c)$. 6. $x-y+1$.

EXERCISE XLV.

A.

p. 188. 1. $9x(x-2)$. 2. $x(ax+b)$. 3. $x^3y^3(pam^2y-qn^3x)$.
4. $(a-1)(b-3)$. 5. $(a+b)(c+d)$. 6. $(a-b)(c-d)$.
7. $(a+c)(a-b)$. 8. $(a+b)(c+2d)$. 9. $(x+3y)(2p+q)$.
10. $(4x+y)(3x-2z)$. 11. $(x-9)(y+4)$. 12. $(p+m)(p-n)$.
13. $(x-2)(x+a)$. 14. $(x+y)(x^2+y^2)$.
15. $(2x^2+3y^2)(3x+2y)$. 16. $(3x^2-2y^2)(2x-3y)$.
17. $(a+b)(a-b+1)$. 18. $(a+2b)(a-2b-3)$.

19. $(px+1)(x-p)$.
20. $(x-y)(x+y-1)$.
21. $(a-b)(a+b+c)$.
22. $(a+b)(c-d)$.
23. $(x+y)(a+b+c)$.
24. $(a-b)(x+y+z)$.
25. $(x+1)(x+2)(2x+3)$.
26. $(a-b)(a+c-d)$.
27. $(a-c)(a+b)(a-b+c)$.
28. (i) $a^2+b^2+c^2$; (ii) $bc+ca+ab$.

p. 189. 31. $-(x-y)(y-z)(z-x)(x+y)(y+z)(z+x)$.

B.

1. $-(x-5y)^2$.
2. $-(x-13y)^2$.
3. $(x^2-14)(x^2+5)$.
4. $(x-19)(x+17)$.
5. $(x^3-24)(x^3+4)$.
6. $(xy-11)(xy+7)$.
7. $a^2b^2(ab+5)(ab-4)$.
8. $a^2(x-6a)(x+5a)$.
9. $(x+17)(x-7)$.
10. $(x-14)(x+13)$.
11. $(x-16yz)(x+15yz)$.
12. $(x-20)(x+17)$.
13. $5(x-9)(x+7)$.
14. $-2(x-11y)^2$.
15. $3(6x-yz)(5x-yz)$.
16. $(x^2-11yz)(x^2-17yz)$.
17. $(x+a)(x-b)$.
18. $(x-2a)(x-4b)$.
19. $(x+c)(x+a+b)$.
20. $(x-3)(x-a-2)$.
21. $(x-a-b)(x+c)$.
22. $(x-a-3)(x+2)$.
23. $(1-11a+11b)(1+10a-10b)$.
24. $(1-4a-4b)(1+a+b)$.

C.

1. $(4x-3)(6-5x)$.
2. $(3x+5yz)(2x-9yz)$.
3. $(3x-5y)(5x+3y)$.
4. $(5x+6)(4x-5)$.
5. $x(6x-5)(4x+3)$.
6. $(2x+7)(9x-5)$.
7. $6(3x-2)(2x-1)$.
8. $(x+8)(3x+7)$.
9. $(6x-7y)(9x+2y)$.
10. $(3x+7)(8x-5)$.
11. $(4x+15y)(6x-5y)$.
12. $(2x^2+5)^2$.
13. $(x+8y)(9x-17y)$.
14. $x^2y(5x-4y)^2$.
15. $(6x+7y)(18x-25y)$.
16. $(3x-5y)(5x+3y)$.
17. $2(3x-7y)^2$.
18. $-(5x-9y)^2$.
19. $-(3x-10y)^2$.
20. $[2(a+b)-(x+y)][a+b-3(x+y)]$.
21. $-(3a-3b+5c)^2$.
22. $(px+q)(rx+s)$.
23. $(px+1)(x+q)$.
24. $(px-1)(x+q)$.
25. $(x+1)(x+2)(2x-1)$.
26. $(x+1)(x+2)(x-3)$.

D.

p. 190. 1. $16(2a-c)(4a-3b+2c)$.
2. $12c(a-2b)$.
3. $(4a-b+c)(2a+3b-5c)$.
4. $(a+3b-c)(a-b+3c)$.
5. $(y-2z+x-a)(y-2z-x+a)$.
6. $(x+5)(x+2)(x-2)$.

p. 190. 7. $(x-2)(x+2)(x+3)$. 8. $(a-c)(a-2b+c)$.
9. $2b(a-b)$. 10. $(a+b)(a-b)(x^2+y^2)$.
11. $(a+b)(a-b)(x+y)(x-y)$. 12. $(a-c)(a+c-2b)$.
13. $(a+b)^2(a-b)$. 14. $(1+y)(1-3x-y)$.
15. $(x+1)(x-1)^2$. 16. $(x+2y)(x-2y+a)$.
17. $(x+1)(x+2)(x-2)$. 18. $(x-y)(x+y)(x^2+y^2)$.
19. $8xy(x^2+y^2)$. 20. $4xy(x+y)^2$.
21. $(b-a)(b+a)(x-y)(x+y)$. 22. $(x-y)(x+y)(2-z)(2+z)$.
23. $(x-3)(x+3)(x^2+4)$. 24. $(x-3)^2(x+3)^2$.
25. $(x-2)(x+2)(x-3)(x+3)$. 26. $(x-3)(x+3)(x-4)(x+4)$.
27. $(x+1)^2(2x+1)$. 28. $(x-1)(x+2)(2x-1)$.
29. $(2x^2-6x+9)(2x^2+6x+9)$. 30. $(x^2-2x-7)(x^2+2x-7)$.
31. $(x^2-10x+1)(x^2+10x+1)$.
32. $(x^2-10xy-y^2)(x^2+10xy-y^2)$.
33. $(2x^2-2xy+3y^2)(2x^2+2xy+3y^2)$.
34. $(2x^2-xy-5y^2)(2x^2+xy-5y^2)$.
35. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

E.

1. $4b(3a^2+4b^2)$. 2. $8(a+b)(19a^2+26ab+19b^2)$.
3. $(3y-x)(19x^2-9xy+3y^2)$. 4. $(x+y)(a-b)(a^2+ab+b^2)$.
5. $(x+y)(a+b)(a^2-ab+b^2)$. 6. $(x-y)(x^2+y^2)(x^2+xy+y^2)$.
7. $(x-1)(x+1)(x^2-x+1)$. 8. $(x-1)^2(x+1)(x^2+x+1)$.
9. $(x-y)(x^2+xy+y^2+x+y)$. 10. $(x+y)^3$.
11. $(x+y)(x^2+4xy+y^2)$. 12. $x(x-y)(2x+y)$.

EXERCISE XLVI.

p. 192. 1. $x-1$. 2. $x-2$. 3. $2x-3$. 4. $3x-2$.
5. $x+5$. 6. $(x-1)(x-2)(x+3)$. 7. $a(x^2-y^2)$; H.C.F. $=a$.
8. $(x+1)(x-1)(x-2)$; H.C.F. $=x-1$.
9. $12ab(a+b)(a-2b)$; H.C.F. $=2a(a+b)$.
10. $a^2b(a+b)(a-b)$. 11. $ab^5c^4z^2(x-y)^3$; H.C.F. $=b^3(x-y)$.
12. $ab(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$ or $ab(a^6-b^6)$.
13. $60a^4b^3(x-y)(x+y)^3$; H.C.F. $=2a^2b(x+y)$.
14. $720a^3b^2c^3(a+b)(a-b)^2(a^2+ab+b^2)$; H.C.F. $=6a^2bc(a-b)$.
15. $12(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)$; H.C.F. $=x-y$.
16. $12x^2y^2(x+y)(x-y)^3(x^2+y^2)$.
17. $216(x-3)^3(x+4)^2$; H.C.F. $=3$.
18. $6x(x+5)(x-7)(x-9)$; H.C.F. $=x(x+5)$.
19. $2x^2y^2(2x-3y)^2(2x+3y)$; H.C.F. $=x(2x-3y)$.

p. 193. 20. $(x-2)(2x+3)(3x-1)$. 21. $(x-a)^2(x+2a)(2x+a)$.
22. $x(x+1)(x-2)(2x+3)$.
23. $2a(2a-b)(2a-3b)(2a+3b)(4a-b)$.
24. $x^2(x-2)(2x-3)(3x-1)$. 25. $ab(a^2+b^2)(a^2+2b^2)(a^2-2b^2)$.
26. $2x^2(x-1)(x-5)(2x-1)$.
27. $(5x-1)(2x+3)(3x-2)(x+3)$; H.C.F. $=5x-1$.
28. $(2x-3y)(2x+3y)(3x-2y)(3x+2y)$.
29. $x(x-1)(x+1)(x-2)(3x-1)(x-7)$.
30. $(x+a+b)(x-a+b)(x+a-b)$; H.C.F. $=x+a-b$.
31. $4abc(a-c)(b-c)$.
32. $(2x-3)(x+1)(x-1)(x-3)$; H.C.F. $=x+1$.
33. $(x-1)(x+1)(x-2)(x+2)$.
34. $(x+y+z)(x-y-z)(y-z-x)(z-x-y)$; H.C.F. $=x+y+z$.
35. $(a+b+c)(a+b-c)(a-b+c)$; H.C.F. $=a+b+c$.
36. $(x-1)(x+1)(4x-1)(3x^2+1)$; H.C.F. $=x-1$.
37. $(x-a)(x+a)^2(x^2+a^2)$.
38. $(xy+a^2)(2x+3y)(2x-3y)$; H.C.F. $=xy+a^2$.
39. $(x-2)(3x+1)(4x^2+3)$.
40. $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)$.
41. $(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.
42. $(2x+3y)(2x-3y)(4x^2+6xy+9y^2)(4x^2-6xy+9y^2)$.
43. $x^2(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.
44. $(a-b)^2(a+b)^2(a^2+ab+b^2)(a^2-ab+b^2)$.
45. $7x+a-b$. 46. $(x-1)(2x-3)$.
47. $(x-a)(x+a)(x-2b)(x+2b)$.

EXERCISE XLVII.

p. 198. 1. $3a^4-26a^3b+37a^2b^2-14ab^3$.
2. $x^4-x^3+11x-15$. 3. $2x^4-x^3-4x^2+44x-21$.
4. $-3x^4+25x^3y-54x^2y^2+46xy^3-16y^4$.
5. $14x^4-33x^3+43x-24$. 6. $x^6-ax^5-a^2x^4+a^4x^2+a^5x-a^6$.
7. $a^7b-a^5b^3+a^3b^5-ab^7$. 8. $x^5-5x^4-5x^3+41x^2-44x+12$.
9. $5x^5-20x^4y+14x^3y^2-3x^2y^3+25xy^4-21y^5$.
10. $3x^5+6x^4-7x^3+20x-8$.
11. $2x^5-15x^4-80x^3+43x^2+35x-15$.
12. $x^4+10x^3+35x^2+50x+24$.
13. $120x^4-26x^3-111x^2+14x+24$.
14. $36x^4-132x^3+157x^2-66x+9$. 15. $6x^5-x^4-8x^3-7x+10$.
16. $25a^5b-44a^3b^3+16ab^5$. 17. $1-5x^4+4x^5$.

p. 198. 18. x^6-1. 19. x^6-64.
20. x^6-2x^3+1. 21. $2-5x^3+3x^5$.

p. 199. 22. $2x^6-5x^5+25x-33$. 23. $6-4x-15x^2+x^3+19x^4-25x^6$.
24. $10x^6-11x^5-2x^4-26x^3+5x^2+x+35$.
25. $abc+(ab-ac+bc)x+(b-c-a)x^2-x^3$.
26. $x^3-y^3-z^3-3xyz$. 27. $8x^3-27y^3+z^3+18xyz$.
28. $x^5-(a+1)x^4+(a-b-1)x^3+(a+b+c)x^2+(b-c)x-c$.
29. $x^4-4x^3+6x^2-4x+1$. 30. $1-5x+10x^2-10x^3+5x^4-x^5$.
31. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.
32. $16x^4+32x^3+24x^2+8x+1$.
33. $32-240x+720x^2-1080x^3+810x^4-243x^5$.
34. $1+12x+60x^2+160x^3+240x^4+192x^5+64x^6$.
35. -42. 36. 64.
37. (i) -28; (ii) -210; (iii) -2; (iv) 7; (v) 20.
38. 25. 39. $2ab$. 40. -30.

EXERCISE XLVIII.

p. 205. 1. $x+5$. 2. $x-2$. 3. $x-8$. 4. $2x+7$.
5. $3x-8$. 6. $7x-9y$. 7. x^2-x+1. 8. x^3-x^2+x-1.
9. $x^4-x^3+x^2-x+1$. 10. $x^4-x^2y^2+y^4$.
11. $x^4+x^3y+x^2y^2+xy^3+y^4$. 12. $9x^2+6x+2$.
13. $2x^2+8$. 14. x^2-3x-5. 15. x^3-3x+2.
16. $x^3+x^2y+xy^2+y^3$. 17. $4x^3-3x^2y-2xy^2+5y^3$.
18. x^3+2x+3. 19. x^2-2x-3. 20. x^2-3x-5.
21. $9x^2+6x+4$. 22. $2x^2+xy-3y^2$. 23. $4x^2-6x-1$.
24. x^2-4x+1. 25. x^2+5x-2. 26. $5x^2+11x+11$.
27. $2x^2-xy+3y^2$. 28. $4x^3-6x^2+5x+7$.
29. $3x^3+5x^2-4x-2$. 30. x^3-2x^2+4x-8.
31. $2x^3+3x^2-x-3$. 32. $3x^3-4x^2+6x-12$.
33. -3. 34. 2. 35. 6. 36. -2.

p. 206. 37. $3x+5$. 38. $-2x-7$. 39. $6x-2$.
40. $x-3$. 41. $-39x+59$.
42. $2x^2+x+5$, $2x+35$; $-2-5x-7x^2$, $54x^3+27x^4$.
43. $4x^2+3x-2$, $-40x-25$; $3+8x+3x^2$, $-13x^3+2x^4$.
44. $(x-2)(x-3)(x+7)$. 45. $(x-1)(x-2)(x-3)$.
46. $(x-5)(x+3)(x-4)$. 47. $(3x+5)(2x-7)(2x+1)$.
48. $(3x+4)(x-2)(2x+3)(x-1)$.
49. $(4x+3)(x-2)(5x+1)(3x-1)$. 50. $(x+1)^3(x-1)^2$.
51. $(x-1)(x-2)(x+5)(2x-3)(3x+2)$.

EXERCISE XLIX.

p. 211. 1. 3. 2. (i) $31\frac{1}{2}$; (ii) -3. 3. (a) $\frac{25}{17}$; (b) $-\frac{4}{3}$; (c) $\frac{8}{3}$.
4. c. 5. (i) $-\frac{1}{3}$; (ii) $\frac{13}{4}$.

p. 212. 6. (a) $-\frac{89}{3}$; (b) $-\frac{37}{12}$; (c) $\frac{250}{77}$. 7. (a) 4; (b) $\frac{12}{5}$.
8. (i) $\frac{1}{6}$; (ii) $\frac{17}{6}$. 9. (i) $\frac{24}{115}$; (ii) $\frac{11}{2}$. 10. (i) 1; (ii) $\frac{107}{4}$.
11. (i) $-\frac{1}{14}$; (ii) $\frac{1}{6}$. 12. (i) $\frac{13}{10}$; (ii) $-\frac{3}{4}$.
13. (i) $-\frac{326}{221}$; (ii) $\frac{7}{10}$. 14. 2. 15. 8·686.
16. (i) $l=\frac{gt^2}{4\pi^2}$; (ii) 0·815.

EXERCISE L.

p. 213. 1. $2x^2$. 2. $\frac{1}{9x^2}$. 3. $\frac{2x+y}{4x+y}$. 4. $\frac{x-10}{x+5}$.
5. $\frac{3x-5}{7x-1}$. 6. $\frac{2x+y}{2y-x}$. 7. $\frac{7y-x}{5y+x}$. 8. $3bc$. 9. $\frac{1}{x+y}$.

p. 214. 10. $\frac{2x-y}{x-2y}$. 11. $\frac{y+2b}{y-2b}$. 12. $\frac{(x-1)^2}{x^2+1}$. 13. $\frac{(x-1)(x+1)}{(x-2)(x+3)}$.
14. $\frac{3bc}{a}$. 15. $\frac{x^2+y^2}{x}$. 16. $\frac{(x+1)(2x+1)}{2x^2}$.
17. $\frac{x^2-3x+1}{x^2+x+1}$. 18. $\frac{x+5}{x+7}$. 19. $\frac{3(3x+1)}{2(2x+1)}$. 20. $\frac{1}{(x+12)^2}$.
21. $\frac{x-4}{x+2}$. 22. $\frac{a-c}{a+c}$. 23. $\frac{2a-c}{2c-a}$. 24. 1.
25. $\frac{x-1}{x-3}$. 26. 1. 27. $\frac{(b-c)(b+c)}{(a-b)(a+b)}$. 28. $\frac{x-y+z}{xy}$.
29. $\frac{x+1}{x-2}$. 30. 1. 31. $\frac{a-2}{a^2-a+1}$. 32. $\frac{x}{x+3}$.
33. 1. 34. $\frac{1}{x-1}$. 35. $x+b$.

EXERCISE LI.

p. 216. 1. $\frac{x^2-4x}{x^2-1}$. 2. $-\frac{1}{a+2b}$. 3. $\frac{a-b}{a+b}$. 4. $\frac{16x}{1-x^2}$.
5. $\frac{1}{(x+1)(x+3)}$. 6. $\frac{1}{(x-2)(x-3)(x+3)}$. 7. $\frac{1}{(x+1)(x-2)}$.
8. $\frac{1}{(x-2)(x-3)}$. 9. $\frac{1}{x^2-a^2}$. 10. $\frac{bx}{c+x}$.

p. 216. 11. $\dfrac{1}{(a+b)(a+3b)}$. 12. $\dfrac{x^2+x+1}{x^2(x+1)}$. 13. $\dfrac{3x-2a}{x(x^2-a^2)}$.

14. $\dfrac{13}{(4x-9)(10x-29)}$. 15. $-\dfrac{2}{(x-2a)(2x-a)(2x+a)}$. 16. 0.

17. $\dfrac{x+c}{(x-a)(x-b)}$. 18. $\dfrac{4y^3}{x(x^2-y^2)}$. 19. $\dfrac{6a}{(x-2a)(x-3a)}$.

20. $\dfrac{5}{(x+1)(x-4)}$. 21. $\dfrac{2x-9}{(x-4)(x^2-1)}$. 22. 0.

23. $\dfrac{2}{(x-1)(x-2)(x-3)(x-4)}$. 24. $\dfrac{4b(b-a)}{(a^2-4b^2)(4a^2-b^2)}$.

25. $\dfrac{2}{(x-3)(x-4)(x-5)}$. 26. $-\dfrac{1}{2(x-1)}$. 27. $\dfrac{x}{2(x-4)}$.

28. $\dfrac{5}{x(x-2)}$. 29. $\dfrac{4(b+c)(a-b-c)}{(a-b+c)(a+b-c)}$. 30. $\dfrac{2}{x^2+xy+y^2}$.

p. 217. 31. $\dfrac{x^2-6x+17}{(x-3)(x-4)(x-5)(x-7)}$. 32. $\dfrac{1}{x-1}$. 33. $\dfrac{2}{2x-1}$.

34. $\dfrac{3x-2}{(2x-3)(x-1)}$. 35. $-\dfrac{15x+2}{(x-1)(x-2)(x+3)}$.

36. $\dfrac{4}{x^2-4}$. 37. $\dfrac{x-1}{x+1}$. 38. $\dfrac{5x-1}{(x^2+1)(2x-3)}$.

EXERCISE LII.

p. 218. 1. $\dfrac{2}{a-b}$. 2. $\dfrac{1}{a+2}$. 3. $\dfrac{1}{1-a^2}$. 4. $\dfrac{6b}{b^2-a^2}$. 5. $\dfrac{1}{1-9x^2}$.

6. 0. 7. 1. 8. $\dfrac{2a^2}{x(x^2-a^2)}$. 9. 0.

10. $\dfrac{2}{x(x+1)(x+2)}$. 11. 0. 12. $\dfrac{6}{(x+2)(x+3)(x+4)(x+5)}$.

13. $\dfrac{x}{(a-x)^2}$. 14. $-\dfrac{48}{x(x-2)(x+2)(x+4)}$.

p. 219. 15. $\dfrac{1}{x(x-a)(x-b)}$. 16. $\dfrac{96}{(x-3)(x+5)(x-7)}$.

17. $\dfrac{2x}{(x-1)(x-2)}$. 18. $\dfrac{2x-3}{(x+1)(x-1)(2x+3)}$.

19. $\dfrac{2}{(x-1)(x-2)(x-3)}$. 20. $\dfrac{5a^2}{a^2+4b^2}$. 21. $\dfrac{a^4+b^4}{a^4-b^4}$.

22. $\dfrac{8}{1-x^8}$. 23. $\dfrac{6}{x^6-1}$. 24. $\dfrac{1}{x(x-1)(x^3+1)}$.

25. $\dfrac{2}{2a-7}$. 26. $\dfrac{2}{(a-1)(2a-1)(2a-3)}$. 27. $\dfrac{1}{4x-3}$.

28. $\dfrac{22}{(x-1)(x-4)(x+4)}$. 29. $-\dfrac{14x^2}{(x+1)(x-2)(2x+1)}$.

EXERCISE LIII.

p. 221. 1. 1. 2. 1. 3. 0. 4. $\dfrac{x(x-3)}{x-1}$. 5. $\dfrac{x-1}{x+3}$.

6. x. 7. $\dfrac{x}{x+a}$. 8. $\dfrac{x+y}{1-xy}$. 9. $\dfrac{a^4}{(a^2-1)(a^4+1)}$.

10. $\dfrac{x^2-y^2}{2}$. 11. 1. 12. $2a(a+b)$. 13. $(x-1)^2$. 14. 1.

15. $-\dfrac{1}{x}$. 16. $\dfrac{x-z}{1+xz}$. 17. $\dfrac{a(b^2-a)}{b(b^2-2a)}$. 18. 1.

19. $\dfrac{x^2+1}{x^2}$. 20. $\dfrac{2-x^2}{2}$. 21. $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}$. 22. 2.

p. 222. 23. $\dfrac{4a-27b}{2a-7b}$. 24. (i) $\dfrac{(a+1)^2}{2a}$; (ii) $\dfrac{5-a}{9a-1}$. 25. $\dfrac{a}{1-2a^2}$.

EXERCISE LIV.

1. $\dfrac{4x^2}{(1-x^2)^2}$. 2. $\dfrac{1}{x+3}$. 3. $\dfrac{3(a+2b)}{(a+b)(2a-b)}$.

4. $\dfrac{4x+1}{2(x-1)(2x-1)}$. 5. $\dfrac{4}{x^2-1}$. 6. $\dfrac{2(x+y)}{x-y}$. 7. 4.

8. 1. 9. 1. 10. 2. 11. 1.

12. $\dfrac{x}{a}$. 13. 1. 14. 1. 15. $\dfrac{2}{x-4}$.

p. 223. 16. $\dfrac{a+b}{b}$. 17. $\dfrac{1}{4}$. 18. $\dfrac{ab}{a^2+b^2}$. 19. $\dfrac{4x^2y^2}{(x-y)(x^3-y^3)}$.

20. $\dfrac{5(x+a)}{(x-2a)^2}$. 21. 2. 22. $\dfrac{a+b}{a-b}$. 23. $\dfrac{3}{4(4x+25)}$.

24. $x-a$. 25. $\dfrac{x^3}{(1+x)(1+x+x^2)}$. 26. $-4(a^2+b^2)$.

27. -1. 28. $\dfrac{a}{b}$. 29. $\dfrac{1}{a+b}$. 30. $\dfrac{1}{b-c}$.

31. 2. 32. $\dfrac{a+b}{ab}$. 33. $\dfrac{y-x}{y+x}$. 34. $\dfrac{4(a+b)x}{x^2-4b^2}$.

p. 224. 35. $\dfrac{x+y}{x-y}$. 36. $\dfrac{x^2+y^2}{2(x^4+x^2y^2+y^4)}$. 37. $\dfrac{1}{x+y}$.

p. 224. 38. $\frac{1}{xy}$. 39. $\frac{b}{c}$. 40. ab. 41. $\frac{1}{a}$.

42. $\frac{10x}{9-x^2}$. 43. $\frac{2a-b}{a-2b}$. 46. $3+\frac{2}{a}$. 47. $\frac{1}{8}\left(x^2-\frac{1}{x^2}\right)^2$.

EXERCISE LV.

p. 226. 1. $x^3-\frac{17}{4}x+\frac{1}{x}$. 2. $\frac{x^3}{2}+\frac{5x}{24}-\frac{1}{6}$.

3. $\frac{x^2}{9}+\frac{17}{240}+\frac{1}{25x^2}$. 4. $\frac{4x^2}{9}-1+\frac{4}{5x}+\frac{4}{25x^2}$.

5. $1+\frac{4x}{3}-\frac{23x^2}{9}-2x^3+\frac{9x^4}{4}$. 6. $\frac{4x^2}{y^2}-\frac{4x}{y}+3-\frac{y}{x}+\frac{y^2}{4x^2}$.

7. $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{2b}{a}+\frac{2c}{b}+\frac{2a}{c}$. 8. $x^4-\frac{13x^3}{6}+x^2-\frac{4x}{9}+\frac{2}{3}$.

9. $\frac{3x^4}{8}+\frac{x^3}{16}+\frac{5x^2}{24}-\frac{x}{72}+\frac{2}{27}$. 10. $\frac{x^2}{9}-4y^2+\frac{z^2}{4}+\frac{xz}{3}$.

11. $\frac{3x^4}{4}-\frac{x^3y}{2}+\frac{xy^3}{4}-\frac{3y^4}{16}$. 12. $x^2+\frac{625}{4x^2}$.

13. $x^2+y^2+z^2+\left(\frac{b}{c}+\frac{c}{b}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)2x+\left(\frac{a}{b}+\frac{b}{a}\right)xy$.

14. $x^3-\frac{7x}{36}+\frac{1}{36}$.

p. 227. 15. $x^3+\frac{x^2}{6}-\frac{14x}{9}+\frac{2}{3}$. 16. $x^4-\frac{2x^3y}{3}+\frac{2xy^3}{27}+\frac{y^4}{81}$.

17. $x^3+\frac{1}{8}y^3+\frac{1}{27}z^3-\frac{1}{2}xyz$. 18. $x^3-\frac{8}{27}y^3+\frac{27}{8}z^3+3xyz$.

19. $-\frac{11}{180}$. 20. $-\frac{11}{30}$. 21. 1. 22. $\frac{1}{3}$. 23. $-\frac{29}{36}$.

EXERCISE LVI.

p. 231. 1. $x^2-\frac{1}{2}x+\frac{1}{3}$. 2. $\frac{1}{2}x^2+\frac{2}{3}x-\frac{3}{4}$. 3. $\frac{2}{3}x^2-\frac{3}{4}x-\frac{1}{2}$.

4. $\frac{1}{2}x^3+x^2-x-1$. 5. $\frac{3x^3}{4}-\frac{x^2}{2}+\frac{1}{3}$. 6. $\frac{3x^2}{2}-\frac{3y^2}{4}$.

7. $\frac{x}{4}-\frac{1}{6}+\frac{1}{9x}$. 8. $x-\frac{2}{x}$. 9. $x+\frac{3}{2x}-\frac{5}{2}$.

10. $\frac{1}{3}x^2+\frac{3}{5}x-\frac{1}{2}$. 11. $\frac{3x^4}{2}-x^3+\frac{x^2}{3}$.

12. $a^2-3ab+\frac{1}{5}b^2$. 13. $x^2-(b+c)x+bc$.

p. 232. 14. $x^2y-2x+2$. 15. $4x^2+2x(y-1)+y^2+y+1$.

16. $x^2-(3a-1)x+9a^2+3a+1$. 17. $1+y-xy-xy^2$.

18. $3x^2+2xy-y+4$. 19. $4x^2-(5a+3b)x+2ab$.

20. $3(a-b)^3+4(a-b)^2(c-d)+12(a-b)(c-d)^2+16(c-d)^3$.

21. $a-2b+c$. 22. x^4-ax+b.

EXERCISE LVII.

p. 237. 1. (i) 7; (ii) 0; (iii) $\frac{3}{52}$; (iv) $-\frac{25}{7}$.

2. (i) $\frac{3}{2}$; (ii) $\frac{3}{5}$; (iii) 7; (iv) $\frac{11}{7}$.

3. (i) $\frac{4}{2}$; (ii) $\frac{5}{3}$; (iii) $\frac{12}{13}$; (iv) $\frac{29}{21}$.

p. 238. 4. (i) 2; (ii) 3; (iii) $\frac{3}{8}$; (iv) $\frac{2}{5}$. 5. (i) 5; (ii) $\frac{2}{9}$.

11. (i) $\frac{15}{17}, \frac{10}{17}$; (ii) $\frac{4}{31}, -\frac{3}{31}$; (iii) $-\frac{cq}{aq+bp}, -\frac{cp}{aq+bp}$.

12. $\frac{x}{y}=\frac{3}{7}$; 3, 7. 13. $\frac{33}{17}$. 14. $\frac{b+c-2a}{c-b}$.

EXERCISE LVIII.

p. 244. 1. $\frac{ap+bq}{p+q}$ shillings. 2. $\left(\frac{120}{x}+\frac{80}{y}\right)$ hours.

3. $\frac{ax}{a+b}$; $\frac{ax}{x+y}$ pence. 4. $\frac{20x}{x+20}$.

5. $\frac{x-z}{y}$ hours. 6. $\frac{xy}{y+z}$. 7. $b(x-y)-\frac{cy}{12}$.

p. 245. 8. £$(12a-73b)$; (ii) $\frac{c}{12a-73b}$. 9. $\frac{pq}{p+q}$.

10. $\frac{mn-bc}{n-b}$ lbs. 11. $\frac{12d-ab}{c}$.

12. $\frac{3ab}{5(a-c)}$ pence per dozen. 13. $\frac{3p(100+r)}{25q}$.

14. $\frac{12(a+b)-ap-bq}{12}$ pence; $\frac{100\{12(a+b)-ap-bq\}}{ap+bq}$.

15. $\frac{ac}{12}-\frac{ab}{20}$; $\frac{100(5c-3b)}{3b}$; $5c=3b$.

p. 245. 16. $\frac{25(a-b)}{c}$ weeks. 17. $\frac{abc}{ab+bc+ca}$.

18. $\frac{xy}{x+y}$ hours; $\frac{xy-yz-zx}{xy}$. 19. $\frac{n(m-p)}{m}$ hours.

EXERCISE LIX.

p. 249. 1. $\frac{25}{2}$. 2. $\frac{1}{2}$. 3. $\frac{21}{5}$.

p. 250. 4. $\frac{4}{7}$. 5. $-\frac{10}{31}$. 6. $-\frac{3}{5}$. 7. 4. 8. $\frac{12}{13}$. 9. $2\frac{1}{2}$.

10. $\frac{11}{13}$. 11. $\frac{1}{5}$. 12. 2. 13. No solution.

14. $-\frac{22}{7}$. 15. 12. 16. -3. 17. $-\frac{4}{3}$. 18. No solution.

19. 14. 20. Satisfied by *any* value of x. 21. $\frac{10}{7}$.

22. 6. 23. $-\frac{125}{38}$. 24. 8. 25. $\frac{31}{6}$. 26. No solution.

27. $-\frac{5}{2}$. 28. -4. 29. 4. 30. 6. 31. -1.

32. $\frac{1}{2}$. 33. 3. 34. 2. 35. $-\frac{5}{4}$. 36. 9.

p. 251. 37. No solution. 38. $\frac{2ab}{a+b}$. 39. $\frac{ab}{a+b}$.

40. $\frac{2a}{3}$. 41. $\frac{2}{a+b}$. 42. No solution.

43. $\frac{ab-cd}{c^2-a^2}$. 44. $\frac{abc}{a^2+b^2+c^2}$. 45. $a+b+c$.

EXERCISE LX.

p. 253. 1. $-\frac{3}{2}, \frac{4}{3}$. 2. $\frac{2}{5}, -\frac{2}{3}$. 3. $-8, -16$. 4. $\frac{2}{7}, -\frac{2}{3}$.

5. $\frac{3}{2}, 0$. 6. $\frac{5}{3}, -\frac{3}{2}$. 7. $-\frac{4}{5}, -1$.

8. $\frac{1}{7}, -1$. 9. $\frac{1}{3}, -3$. 10. $\frac{2}{7}, 4$.

p. 254. 11. $\frac{23}{7}, \frac{9}{7}, \frac{7}{8}$. 12. $\frac{19}{3}, 3, -\frac{3}{20}$. 13. $\frac{8}{3a}, \frac{16}{3a}, -\frac{16}{5a}$.

EXERCISE LXI.

p. 256. **1.** $\frac{13}{12}, \frac{7}{12}, \frac{1}{3}$. **2.** $x=2$; £5 5*s*. **3.** $3\frac{2}{3}$ miles. **4.** 20.

5. 60 lbs. **6.** 24 lbs. at 1*s*. 8*d*. a lb., 48 lbs. at 1*s*. 6*d*. a lb.

7. $3\frac{3}{7}$ miles per hour.

p. 257. **8.** 2600 yds. **9.** $88\frac{1}{3}$ miles. **10.** 46 miles from the start.

11. $12\frac{1}{2}$ miles from the start.

12. Rode $1\frac{1}{2}$ hours; walked $3\frac{1}{2}$ hours.

13. 11. **14.** $\frac{4}{15}$. **15.** $\frac{20}{21}$. **16.** $\frac{4}{5}$.

p. 258. **17.** $\frac{7}{9}$. **18.** $\frac{37}{73}$. **19.** 58 miles. **20.** 2340.

21. 235 yds. per min. **22.** 21 ft. $10\frac{1}{2}$ in.

23. £11 each. **24.** £150. **25.** £3600.

p. 259. **26.** 25 shillings. **27.** *A*, $12\frac{1}{2}$ acres; *B*, $11\frac{2}{3}$ acres.

28. $1\frac{1}{4}$ miles per hour. **29.** £400 at 80; £450 at 90.

30. Labour, £4. 16*s*.; materials, £7. 4*s*. **31.** £31, £60.

32. 37. **33.** £342. 10*s*. at $3\frac{1}{2}$ %; £657. 10*s*. at 5 %.

34. £1060. **35.** $3\frac{23}{29}$ %.

EXERCISE LXII.

p. 265. **1.** (i) $c=0$; (ii) $a=0$; (iii) $af+bg=c$; (iv) $b=0$; (v) $a=c=0$.

2. (i) $-\frac{a}{b}$; (ii) $ax+by=0$; (iii) $a(x-f)+b(y-g)=0$.

3. $-\frac{c}{a}, -\frac{c}{b}$. **4.** (ii) $\frac{x}{a}+\frac{y}{b}=1$.

p. 266. **5.** $\frac{x}{a}+\frac{y}{b}=1$; $-\frac{x}{a}+\frac{y}{b}=1$; $-\frac{x}{a}-\frac{y}{b}=1$; $\frac{x}{a}-\frac{y}{b}=1$.

8. (i) $\frac{a}{b}=\frac{a'}{b'}$; (ii) $aa'+bb'=0$.

9. (i) $2x-5y+14=0$; (ii) $2x-5y+26=0$;
(iii) $5x+2y-20=0$; (iv) $5x+2y+40=0$.

10. (i) $2x+y-5=0$; (ii) $3x-2y-22=0$; (iii) $x-y+2=0$;
(iv) $3x-2y+1=0$; (v) $5x-7y+21=0$; (vi) $6x+7y+14=0$.

12. (i) $y=2x-2$; (ii) $x+y=7$; (iii) $3y-x-9=0$;
(iv) $a(x-3)+b(y-4)=0$.

13. (i) $x+2y+3=0$; (ii) $x-y-3=0$; (iii) $3x+y-1=0$;
(iv) $b(x-1)-a(y+2)=0$.

p. 267. **14.** (i) 210; (ii) 160. **15.** (i) 174; (ii) 208.

16. (i) $-7{\cdot}7$; (ii) 268. **17.** (i) 57; (ii) 211.

18. -102. **19.** 1·64, 1·30. **20.** $-2{\cdot}8$, $-6{\cdot}0$.

21. 4·2, -64. **22.** 5·9, $-6{\cdot}6$.

EXERCISE LXIII.

p. 273. **1.** 579 gallons ; 99 cu. ft. **2.** 191 lbs. ; 39·5 kg.

3. (i) 12*s.* 7*d.* ; (ii) 14*s.* $7\frac{1}{2}$*d.* ; (iii) $14\frac{1}{4}$ lbs. ; (iv) $15\frac{3}{4}$ lbs.

4. $y=\frac{4}{5}x$; £411 ; 480 rupees.

5. 36·1 % ; 28·4 % ; £21. 18*s.* ; £29. 9*s.*

6. £2. 8*s.* 6*d.* ; £2. 10*s.* 4*d.* ; £851. 6*s.* ; £878. 16*s.*

p. 274. **7.** $y=\frac{81}{55}x$; 3*s.* $7\frac{1}{2}$*d.* ; 5*s.* $3\frac{1}{2}$*d.* ; 1*s.* 4*d.* ; 4*s.* 1*d.*

8. 6 ft./sec. ; 73 ft./sec. ; 15 m./hr. ; 40 m./hr.

9. 12° R. ; 64° R. ; 84° F. ; 149° F. **10.** 60 ; 87.

11. $a=3100$, $b=17$; 4*s.* $2\frac{1}{2}$*d.* ; 61 guests.

p. 275. **12.** £160 to £570 ; £600 to £645 ; £700 to £750.

13. $P=(0{\cdot}31)W+91$.

EXERCISE LXIV.

p. 280. **1.** 0, 1. **2.** 0, 3. **3.** 0, 4. **4.** $a, -b$.

5. $0, \frac{b}{a}$. **6.** $0, -\frac{2}{3}$. **7.** $+\frac{2}{3}, -\frac{2}{3}$. **8.** $0, -\frac{4}{9}$.

9. $-\frac{1}{2}, \frac{2}{3}$. **10.** 0, 1, −1. **11.** $0, -a, -b$. **12.** $0, -\frac{4}{3}, \frac{5}{4}$.

13. 0, −1, 5. **14.** 1, 2, 3, 4. **15.** $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{5}, \frac{5}{6}$.

16. $0, +a, -a$. **17.** −2, 3. **18.** 4, 6. **19.** 1, 4.

20. $-5, -\frac{1}{3}$. **21.** $5, \frac{1}{5}$. **22.** $\frac{5}{2}, \frac{15}{4}$. **23.** −1, 2, −2.

24. 3, 1, −1. **25.** $\frac{3}{2}, 2, -2$. **26.** $x^2-5x+6=0$.

27. $x^2+x=6$. **28.** $x^2-x=6$. **29.** $x^2+5x+6=0$.

30. $x^2=ax$. **31.** $x^2+ax=0$. **32.** $x^2-8ax+15a^2=0$.

33. $x^2-2ax=15a^2$. **34.** $x^2+2ax=15a^2$.

35. $x^2+8ax+15a^2=0$. **36.** $x^2-x(a+b+c)+ab+ac=0$.

37. $x^2+x(a-b+c)-ab+ac=0$. **38.** $x^2-2ax+a^2=4b^2$.

39. $15x^2-8x+1=0$. **40.** $8x^2-2x=3$.

41. $c^2x^2-c(a+b)x+ab=0$. **42.** $c^2x^2-2cax+a^2=b^2$.

43. $(a^2-b^2)x^2-2(a^2+b^2)x+a^2-b^2=0$.

44. $x^3-(a+b-c)x^2+(ab-bc-ca)x+abc=0$.

45. $x^3-(a-b-c)x^2+(bc-ca-ab)x-abc=0$.

46. $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc=0$.

47. $x^3-10x^2+31x-30=0$. 48. $x^3-19x+30=0$.

49. $x^3+6x^2-x-30=0$. 50. $x^3+10x^2+31x+30=0$.

51. $2x^2-3xy-6x+9y=0$.

52. $2x^2-7xy+5y^2-13x+31y+6=0$.

53. $3x^2-xy-4y^2-31x+39y+20=0$.

54. $12x^2-41xy+35y^2-5x+8y-3=0$.

EXERCISE LXV.

p. 282. 1. $0, \frac{1}{2}$. 2. $1, 2$. 3. $3, -7$. 4. $-3, -8$. 5. $1, -\frac{1}{2}$.

6. $3, \frac{1}{2}$. 7. $\frac{7}{2}, \frac{7}{8}$. 8. $1, -\frac{5}{3}$. 9. $-5, \frac{1}{3}$. 10. $-1, \frac{6}{5}$.

11. $6, -\frac{3}{2}$. 12. $\frac{a}{6}, \frac{5a}{3}$. 13. $7, 2$. 14. $5, \frac{7}{2}$.

15. $\frac{3}{2}, -7$. 16. $5, 5$. 17. $11, -8$. 18. $10, -21$.

p. 283. 19. $3, -3$. 20. $3, -3$. 21. $0, 2$. 22. $0, -\frac{13}{6}$.

23. $\frac{1}{2}, \frac{5}{3}$. 24. $\frac{1}{3}, -\frac{3}{2}$. 25. $\frac{5}{4}, -\frac{4}{3}$. 26. $\frac{7}{6}, -\frac{1}{6}$.

27. $\frac{2}{9}, -\frac{2}{3}$. 28. $\frac{3}{2}, -\frac{4}{5}$. 29. $\frac{2}{3}, -\frac{3}{8}$. 30. $-2, -\frac{21}{8}$.

31. $\frac{2}{3}, -1$. 32. $\frac{7}{5}, \frac{7}{5}$. 33. $8, -\frac{17}{3}$. 34. $11, -\frac{1}{2}$.

35. $3, -\frac{1}{3}$. 36. $\frac{3}{2}, -\frac{2}{3}$. 37. $\frac{5}{2}, -\frac{2}{5}$. 38. $\frac{5}{3}, -\frac{2}{3}$.

39. $\frac{3}{5}, -\frac{5}{3}$. 40. $1, -\frac{1}{2}$. 41. $5, -\frac{3}{2}$. 42. $2, \frac{1}{2}$.

43. $1, 1$. 44. $\frac{a-b}{a+b}, -\frac{a+b}{a-b}$. 45. $2, -2$.

46. $b, -b$. 47. $2a, 3a$. 48. $a, -b$.

49. $2, \frac{b}{a}$. 50. $2b, \frac{2}{a}$. 51. $a, b-a$.

52. $a+b, a-b$. 53. $2b+a, 2b-a$. 54. $a, a-b$.

55. $a-c, -2a$. 56. $\frac{a}{b}, -\frac{b}{a}$. 57. $a, -\frac{ab}{a+b}$.

EXERCISE LXVI.

p. 287. **1.** 16. **2.** $\frac{25}{4}$. **3.** $\frac{1}{4}$. **4.** $\frac{1}{9}$. **5.** $\frac{1}{36}$. **6.** $\frac{9}{25}$.

7. $\frac{b^2}{4a^2}$. **8.** $\frac{9a^2}{4}$. **9.** $(x-5)(x-7)$. **10.** $(x+6)(x+8)$

11. $(x-12)(x+2)$. **12.** $(x+8)(x-6)$.

13. $(x-12)(x+3)$. **14.** $(x+22)(x-11)$.

15. $(x+2)(2x-1)$. **16.** $(x-2)(3x+5)$.

17. $(2x+3y)(4x-y)$. **18.** $(10x+11y)(x-y)$.

19. $(x+a)(x+b)$. **20.** $(x-a)(x-b)$.

21. $1, -\frac{7}{5}$. **22.** $3, \frac{5}{3}$. **23.** $\frac{1}{2}, -4$. **24.** $\frac{3}{4}, -\frac{1}{2}$.

25. $\frac{1}{3}, -\frac{2}{9}$. **26.** $\frac{3}{2}, -\frac{5}{6}$. **27.** $\frac{2}{3}, -\frac{7}{6}$. **28.** $\frac{4a}{3}, -8a$.

29. $\frac{3a}{5}, -\frac{a}{2}$. **30.** $2a, \frac{7a}{2}$. **31.** $\frac{a}{2}+b, \frac{a}{2}-b$. **32.** $2a+b, -b$.

33. $a+2, a-4$. **34.** $2a+5b, 2a-b$. **35.** $a, -\frac{4a}{3}$.

EXERCISE LXVII.

p. 290. **1.** $\frac{8}{13}, -\frac{2}{3}$. **2.** $\frac{7}{13}, -\frac{5}{7}$. **3.** $\frac{16}{3}, -\frac{9}{4}$.

4. $\frac{22}{7}, -\frac{7}{22}$. **5.** 5·3, −1·3. **6.** 3·7, −1·3.

7. 1·7, −0·4. **8.** 1·75, 0·95. **9.** 0·04, −2·04.

10. 1·12, $0{\cdot}2\dot{6}$ (or 0·27 approx.).

11. 0·175, $0{\cdot}5\dot{6}$ (0·57 approx.).

12. −·23, $0{\cdot}01\dot{5}71428\dot{8}$ (or 0·016 approx.).

13. $7, -\frac{2}{3}$. **14.** $\pm\frac{4}{3}$. **15.** $3, \frac{11}{2}$. **16.** $1, -\frac{6}{7}$.

17. $3, -\frac{5}{2}$. **18.** $2, -3$. **19.** ± 5. **20.** $4, \frac{7}{3}$.

21. $2, -4$. **22.** $0, -\frac{5}{3}$. **23.** $4, -7$. **24.** No solution.

25. 6, 11. **26.** $0, \frac{4}{5}$. **27.** $7, \frac{7}{2}$. **28.** $8, -9$.

p. 291. 29. $\frac{1}{5}$, $\left(-\frac{4}{3}\text{ is not a solution}\right)$. 30. 6, -1. 31. $-2, -\frac{7}{5}$.

32. $-\frac{4}{3}, -\frac{17}{3}$. 33. $2, \frac{2}{21}$. 34. $-1, \frac{13}{20}$.

35. $\frac{1}{2}$, $\left(\frac{2}{3}\text{ is not a solution}\right)$. 36. $8, \frac{4}{3}$. 37. $1, \frac{3}{17}$.

38. $-\frac{1}{5}, -\frac{7}{5}$. 39. $5, \frac{1}{4}$. 40. $4, \frac{7}{3}$. 41. $4, \frac{9}{8}$.

42. $13a, -a$. 43. $a, \frac{b}{5}$. 44. $a, 2b$.

45. $-1, \frac{b-c}{b-a}$. 46. $\frac{a^2}{b}, \frac{b^2}{a}$. 47. $a-1, a-2$.

48. $\frac{a+3b}{4}, \frac{a+5b}{6}$. 49. $a+b, 2a-b$. 50. $2a+c, -(a+2c)$.

51. $\frac{a}{1-a}, \frac{a+1}{a}$. 52. $\frac{a-b}{a+b}, \frac{a+b}{a-b}$. 53. $\frac{1}{1-k}, \frac{k^2-2}{1-k}$.

54. $\frac{a-b}{a}, \frac{3b}{a-b}$. 55. $7-a$. 56. $\frac{3a-8}{a-3}$.

EXERCISE LXVIII.

p. 297. 1. 14, 15. 2. 8, 9. 3. 11, 13. 4. 10, 11, 12.

5. 8, 9. 6. 7, 9, 11. 7. 39 or -37. 8. 11 or -14.

9. 11 or $\frac{7}{3}$. 10. 5, 45 or $-\frac{50}{9}, \frac{500}{9}$. 11. 4, 6.

12. 7, 5 or $-5, -7$. 13. 36; (21 is inapplicable).

14. 13, 15. 15. 15, 35. 16. 20, 25 or $-25, -20$.

17. 3, 5, 7, 9, 11, 13. 18. 24 ft. by $4\frac{1}{2}$ ft.

p. 298. 19. 30. 20. *A*, 15 days; *B*, 10 days; *C*, 6 days. 21. 10.

22. 15 ft., 16 ft. 23. $4\frac{2}{5}$ miles, 22 minutes. 24. 23 or 32.

25. 121 yds. 26. 9*d.* a dozen. 27. $4\frac{1}{2}$*d.* a dozen.

28. 15. 29. 25 days.

p. 299. 30. $12\frac{1}{2}$ miles per hour. 31. $4\frac{1}{3}$ miles per hour.

32. $27\frac{1}{2}$ miles per hour. 33. 4*s.*, 2*s.* 6*d.*

34. 30 sacks at 10*s.* each.

35. $22\frac{1}{2}$ miles per hour, $17\frac{1}{2}$ miles per hour.

36. 30 secs. 37. 18 miles, 12 miles.

TEST PAPERS.

1.

2. (i) 9 ; (ii) 49 ; (iii) 12. 3. $3a$. 6. (i) $3x=4y$; (ii) 4, 12 miles.
7. $x=y+z$.

2.

2. (i) 2 ; (ii) 8, 9. 4. (i) $y^2(x^3+5)$; (ii) $(x+2y)(x+2y+3)$; (iii) $(x+y)^2$.
5. 499. 7. $[a-(b+c)]^3$; 8. 8. (i) 10000 ; $15(a+b+c)$.

3.

1. $12x+5y$. 2. (i) $5x$; (ii) $5y-10$. 3. $x^2y-4xy^2+4y^3$, 9.
5. (i) 4 ; (ii) 6. 6. 2. 8. $s^2=3r^2-3q^2+p^2$.

4.

2. $30x^7y^7$. 3. (i) $3a+3b+3c$; (ii) $9a+18b-27c$.
4. (i) $12x^2+x-6$; (ii) $30x^2+71x+42$; (iii) $40x^4-33x^2+5$.
5. (i) $16a^3$; (ii) $10ab$; (iii) $8b^3$. 7. H.C.F. $=2a^2b^3c^2$; L.C.M. $=8a^4b^6c^6$.
8. A, £300 ; B, £150.

5.

1. (i) 6 ; (ii) 18 ; (iii) 23. 2. $2y^3-9y^2+16y-14$.
3. $8abc$. 4. $12xy^3+13y^4$.
5. (i) $2ax-2by$; (ii) $3a^2b+3ab^2$; (iii) $432x^2$; (iv) $(x+y)^2$.
6. $(x^2-x+1)(a+b+c)$.
7. (i) $(x-5y)(x+3y)$; (ii) $x^2(x+6)(x-4)$; (iii) $(a-b)(a+b-c)$; (iv) $4(a+2b)^2$; (v) $(x+a)(x-a)(x-b)$.

6.

1. $(a^2+b^2)(x^2+y^2)$. 2. 4,471,740,000. 3. $2a^2+6$.
4. (i) $(x-15)(x+8)$; (ii) $(x-12y)(x+8y)$; (iii) $(2x+3y)(3x-2y)$; (iv) $7y(x-2y)(x+2y)$; (v) $(a+2b+3)(a-2b-3)$; (vi) $2(5-2x)(25+10x+4x^2)$. 5. $(a-b)(a-1)(b-1)$.
6. (i) $(x-1)(x+1)(x+2)$; (ii) $xy(x^2+y^2)(x^4-x^2y^2+y^4)$; (iii) $4(a+1)(2a+3)(2a-1)$; (iv) $x^2y^2(2x-y)(2x+y)(4x^2+y^2)$.
7. (i) $(a+1)(a^2+a+1)$; (ii) $a(a-1)(a+1)^2(a^2+a+1)$. 8. £$(2bd)$.

7.

1. $(5x-6)(3x+2)$. 2. (i) 10 ; (ii) 67 ; (iii) 4. 3. £33, £47.
5. (i) $x(x+7)(x-6)$; (ii) $(2x-5yz)(2x+3yz)$; (iii) $(x-p+q)(x-p-q)$; (iv) $2(3a-b)(3b-2a)$; (v) $(x^2-5xy+y^2)(x^2+5xy+y^2)$.
6. 3. 7. 59, 28, 17, 11, 7, 4. 8. £900.

8.

2. (i) $4(2x-3)(3x+4)$; (ii) $(1+2x+y)(1-2x-y)$; (iii) $3(x-1)(x+1)$; (iv) $xy(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$; (v) $3(x-5)(1-x)$.
3. (i) 7; (ii) 12; (iii) 3, 8.
4. $a+2b+c$; the equation becomes an identity if $a=2b$.
5. 74.
7. $x=a+1$, $y=b-2$.
8. x pence.

9.

1. (i) $(7a-b-c)(3a+b+c)$; (ii) $(x+9y)(7x+3y)$; (iii) $2x(x-3)(x^2+3)$; (iv) $x^2y^2(3x-2y)(9x^2+6xy+4y^2)$.
2. (i) 2; (ii) 3, 4, 1.
3. (i) $2a^3-7a^2-11a+16$.
5. $16m-4n$.
7. $A=a+2b+c$, $B=a-c$, $C=a-2b+c$.
8. $4c(6a+6b-c)$.

10.

2. (i) $a \div b \div c \times d$; (ii) $a-b-c+d$.
3. (i) $\dfrac{5b^3y^3}{2ax}$; (ii) $\dfrac{ab(a+b)^2}{2(a-b)}$; (iii) $x^2-2x-24$.
4. (i) 9; (ii) 7, 5; (iii) $a+2b$.
5. (i) $\dfrac{31-x}{10}$; (ii) $\dfrac{3(x+9)}{2(x-1)}$; (iii) $\dfrac{3(x-3)}{x-5}$; (iv) $(x-8)(x-9)^2$.
6. $a+b+c$.
7. $\dfrac{bx+ay+by}{b}$.
8. A, £48; B, £33.

11.

2. $983^2=966289$, $984^2=968256$, $985^2=970225$.
3. 1932570; 972158532361.
5. (i) 13; (ii) 65; (iii) 65: $65^2=63^2+16^2=33^2+56^2$.
6. £80.
7. 2, 1, 3, 5.
8. 65; 25, 23, 17.

12.

1. (6, 10), (2, 2), (10, 6); 0·24 sq. in.
2. (i) $(a-b)$; (ii) $(x+3a)(x+4a)$; (iii) $a+b$; (iv) $(3x-4y)(4x+3y)$.
3. $x=b+c-a$, $y=c+a-b$, $z=a+b-c$.
4. ". . . the cubes of the first and last is equal to 4 times the sum of the cubes of the remaining numbers."
5. (i) 1; (ii) $2(2a+b)(a+2b)$.
6. $3a-2b$: each expression $=1$, when $x=3a-2b$.
7. 1296 sq. ft.
8. 26 gallons, 13 gallons.

13.

3. (i) $2a+2b-2c$; (ii) $-105c$; (iii) $24-5a$. 4. $-12b$.
5. (i) -2; (ii) $-1, 3, -5$; (iii) $(a+b), a, -b$. 6. $2a+b, a-2b$.
7. 4, 5, 12. 8. 12 men, 10 days.

14.

3. $x^2+4y^2+25z^2-4xy-10xz+20yz$. 4. (i) -5; (ii) -21; (iii) $3, -5$.
5. $+2x^2y+3x^2z-4xy^2-9xz^2+12y^2z+18yz^2-12xyz$.
6. $x^3+(a+b+c)x+(bc+ca+ab)x+abc$; $1-2a-5a^2+6a^3$.
7. (i) -1; (ii) 19.
8. $\frac{a}{u-v}$. (i) A and B are travelling in opposite directions along the same road, A at 4 miles an hour, B at 3 miles an hour. At noon they are 4 miles apart. When do they meet? *Ans.* At 2 p.m.
(ii) A and B are travelling in the same direction along the same road, A at 3 miles on hour, B at 4 miles an hour. At noon B is 2 miles in front of A. When did B pass A? *Ans.* At 10 a.m.

15.

1. (i) -6; (ii) 0. 2. (i) $x+2y$; (ii) $5x-3$.
3. (i) $(2x+3)(6x-5)(3x-1)$; (ii) $(4x^2-a^2)^2(x^2-b^2)^2$.
4. (i) $7xy-8$; (ii) $(x+2y)(x-2y)(8x+5y)$; (iii) $5x-3y-2$.
5. (i) $-1, 5$; (ii) $3, 5, -2$.
6. $x=a-b$, $y=\frac{a}{b}$: the equations become identical if $a+b=0$.
7. 39 yds. at 6*s.* 3*d.* a yd., 31 yds. at 5*s.* 5*d.* a yd. 8. £30, £60.

16.

1. (i) $4x^5+5x^4-8x^3-14x^2-4x+1$; (ii) $36x^4+60x^3-47x^2-60x+36$; (iii) $x^5-10x^3+20x^2-15x+4$. 3. 59
4. (i) $10x^5+3x^4-2x^3-7x^2-4x+4$
$$=(2x^3+x^2-x-2)(5x^2-x+2)+(-4x+8);$$
(ii) $4-4x-7x^2-2x^3+3x^4+10x^5$
$$=(2-x-9x^2-3x^3)(2-x+5x^2)+(45x^4+25x^5).$$
5. (i) -22; (ii) 4. 6. 235.
8. A, 1560 shells, 97 hits; B, 520 shells, 163 hits.

17.

2. (i) $\frac{x+3}{x+5}$; (ii) $\frac{1}{(x+2)^2}$; (iii) $\frac{(x-1)(x+1)}{(x+3)(x-2)}$; (iv) 1. **3.** (i) $x-7$.

4. (i) $-\frac{1}{2y(x+3y)}$; (ii) $\frac{2(x-1)}{(x-3)(x-5)}$. **6.** (i) $34\frac{1}{2}$; (ii) $5\frac{2}{3}$; (iii) 9, 12.

7. $-ab$: the equation becomes an identity if $a=b$. **8.** 24.

18.

2. (i) $12\frac{2}{15}$; (ii) 8.

3. (i) 1; (ii) $\frac{4-x}{(2x-3)(3x-2)}$; (iii) $1-x$; (iv) $\frac{x^2-ab}{x(a-b)}$.

4. (i) $(m-n)$; (ii) $\frac{2}{m-n}$. **6.** 1024·6, -1·033, -1·000.

7. $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$$=[a^2(b-c)+b^2(c-a)+c^2(a-b)](a+b+c).$$

8. 80, £32.

19.

5. (i) $\frac{25(a-b)}{c}$ weeks; (ii) $\frac{25(a-b)}{c(c-1)}$ weeks.

6. (i) $\frac{2xy}{x^2+y^2}$; (ii) $\frac{a(2a+b)}{(2a-b)^2}$. **7.** (i) $\frac{2a}{3}$.

8. (i) $\frac{abc}{bc+ca+ab}$; (ii) $\frac{bc(a-m)}{bc+ca+ab}$.

20.

1. (i) $x=-y-\frac{6ck}{6-c}$: (ii) the equations are inconsistent if $c=6$ and k is not zero; they are not independent if $c=6$ and $k=0$: (iii) (α) $x=-y=-36006$; (β) $x=-y=-0$·36006.

2. The value $\frac{3}{2}$ of x is *not* a solution.

3. (i) $\frac{x}{(x-1)(x-2)(x-3)}$; (ii) $\frac{x^2}{x^2-1}$; (iii) 4.

4. (i) $\frac{88(60x^2-y^2)}{3x^2}$ yds.; (ii) $\frac{acy-pqb}{bx}$ days.

5. (i) $3\frac{1}{4}$; (ii) 3; (iii) $\frac{ab}{6(2a-3b)}$; (iv) -3.

7. a. **8** 6 hrs., 10 hrs., $112\frac{1}{2}$ mi.

21.

2. $\pm\frac{1}{2}, -\frac{3}{2}$; (ii) $-1, 2, \pm 3$; (iii) $\frac{6}{5}, \frac{10}{3}$; (iv) $\frac{1}{6}, \frac{7}{8}$.
3. (i) $x^3-19x+30=0$; (ii) $30x^3-19x^2+1=0$;
(iii) $(a^2-b^2)x^2-2acx+c^2=0$.
4. $(3x+2)(4x-5)(x+2)(2x-1)$; $-\frac{2}{3}, \frac{5}{4}, -2, \frac{1}{2}$.
5. $2y^3+15y^2+33y+13$. 6. (i) $\frac{2ax}{a^2-b^2}$ hours. 7. $\frac{2a^2}{1-a^2}, \frac{2b^2}{1-b^2}$
8. $6\frac{2}{3}$ miles per hour, 10 miles per hour.

22.

1. (i) 108, -48; (ii) 1·08, 0·72; 7, $\frac{9}{5}$.
2. $a=5$ or $-\frac{11}{3}$; the other values of x are -12 and $\frac{16}{3}$ respectively.
3. $x=\frac{4}{3}$ or $-\frac{2}{3}$; the quotients are $2a^2+2a+8$, $2a^2+2a+2$ respectively
4. 144. 5. (i) $\frac{1}{x-2}$; (ii) 1; (iii) $\frac{4(x+y)}{2x+y}$.
6. (i) 13, $\frac{5}{2}$; (ii) $-\frac{3}{2}$ (5 is *not* a solution); (iii) $\frac{1}{8}, -\frac{3}{4}$; (iv) $-\frac{p}{q}, -\frac{qb}{pa}$.
7. 36, 77, 85 inches. 8. 13 miles per hour.

23.

1. 6, 2·5, 6·5. 2. (i) 45; (ii) 10; (iii) 9.
3. (i) $9y-5x=84$, $8x-y=13$; (ii) $9y-5x=17$, $8x-y=80$; (iii) 11, 8.
6. (i) £1. 19*s*. 6*d*.; (ii) 17 dollars 10 cents. 7. 39 in.

24.

1. (i) A parallelogram; (ii) $(2, -2)$, $(2, -2)$; (iii) the diagonals of a parallelogram bisect one another.
2. Area of each $=21$ units.
3. (i) $x+2y-2=0$, $2x-y+6=0$, $4x+3y+2=0$; (ii) $(-2, 2)$;
(iii) $OP=OQ=OR=5$.
4. (ii) Area of each $=90$ units; (iii) triangles on the same base and between the same parallels are equal in area.
5. $-40°$ C. or $-40°$ F.

6. (i) The relation is $P = N\left(\frac{3}{20} - b\right) - a$; (ii) 2000 guests, £62. 10*s*., $a = 50$.

7. 5 hrs. 36 mins. a.m.; $11\frac{1}{2}$ miles.

25.

1. (i) $\left(-4, -\frac{11}{2}\right)$, $\left(-\frac{1}{2}, \frac{5}{2}\right)$, $\left(-\frac{3}{2}, 10\right)$;
(ii) $x - 4y + 26 = 0$, $7x + 3y - 4 = 0$, $8x - y + 22 = 0$; $G(-2, 6)$;
(iv) the medians of a triangle meet in a point.

2. (i) (3, 4), $\left(0, -\frac{7}{2}\right)$; (ii) $5x - 2y - 7 = 0$, $5x - 2y + 24 = 0$; (iv) the straight line bisecting two sides of a triangle is parallel to the third side.

3. (i) $2x + 3y + 15 = 0$, $x - 5y + 40 = 0$, $19x - 4y - 150 = 0$;
(ii) $3x - 2y - 10 = 0$, $5x + y - 21 = 0$, $4x + 19y - 35 = 0$; (iii) (4, 1);
(iv) the perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

4. 2·4.

5. (i) 0·8 and 0·5 fr. per litre; (ii) 1*s*. 2*d*. per pint.

6. $m = 0{\cdot}174$, $p = 0{\cdot}036$.

7. (i) $\left(\frac{1}{k}, 3 - \frac{2}{k}\right)$; (ii) they increase numerically, and by making k small enough, they can be made as great, numerically, as we choose; (iv) parallel straight lines.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

Characteristics of the Series.

The "New Geometry for Schools" published in 1903, closely follows the treatment of the subject advocated by the Mathematical Association and in the syllabus formulated by the University of Cambridge.

The "New Geometry for Junior Forms" was prepared primarily as a text-book for the Junior Locals (Pass Examinations of Oxford and Cambridge. A supplement to it was issued shortly afterwards under the title of "A New Geometry for Senior Forms." The two volumes together form "a Course for Students who desire a competent knowledge of Geometry" (*Plane and Solid*) "without those higher parts usually called 'Modern Geometry.'"

To meet the requirements of various important examining bodies the "New Geometry for Junior Forms" has now been supplemented by two additional volumes, one taking the student to the end of Euclid, Book IV., the other to the end of Book VI.; and a separate volume has been issued under the title of "A New Geometry for Middle Forms," containing the equivalent of Euclid I.-IV.

The "New Geometry for Junior Forms" may be used with any of the other new issues: for although the course of work is divided differently in "Middle Forms," the numbering of sections, figures, experiments and theorems is the same throughout. The scope of each part of the "New Geometry" is as follows:—

Part I. is equivalent to Euclid, Book I.
„ II. „ Euclid III., 1-34, and the easy parts of IV.
„ III. „ Euclid II., III., 35-37, and the harder parts of IV.
„ IV. consists of an Algebraical treatment of Ratio and Proportion, and the equivalent of Euclid VI.
„ V. Solid Geometry, including among other matters, the substance of Euclid, Book XI.

Parts I.-IV. in the new issues are now divided into two sections (*a*) practical, (*b*) theoretical, and followed by examination papers—which are the latest obtainable—demanding for their solution a knowledge of all the preceding parts.

LONDON: MACMILLAN AND CO. Ltd.